§1. Introduction to QFT : PART 2.

<u>Recall</u> : $\underbrace{\int e^{-S/\hbar} \quad \text{path integral.}}$

Given volume form.

$$\Omega = e^{f(x)}\, dx^1 \wedge \cdots \wedge dx^n .$$

Consider integration map $\int : \underbrace{\mathcal{A}}_{\text{certain functions on } \{x_i, \theta_i\}_i} \longmapsto \mathbb{R} \quad (\mathbb{C})$

$$\theta_i \theta_j = -\theta_j \theta_i$$

If we consider integrate over $\mathbb{R}^n$, $\int_{\mathbb{R}^n}$ picks only the components only without θ_i' (PV^0)

$$\int_{\mathbb{R}^n} : f(x) \longmapsto \int_{\mathbb{R}} f(x)\, \Omega$$

defined on Δ-homology $\quad \Delta : \mathcal{A} \longmapsto \mathcal{A}$, BV-operator.

$$\left(\Delta = \sum_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial \theta_i} + \sum_i (\partial_i f) \frac{\partial}{\partial \theta_i} \right)$$

<u>Eg</u> $\quad \int_{\mathbb{R}^n} \Delta(\varphi^i(x)\, \theta_i)\, \Omega = 0$

$\Rightarrow$ (IBP) $\quad \int_{\mathbb{R}^n} \left(\sum_i \partial_i \varphi^i + \sum \varphi^i\, \partial_i f \right) e^f\, d^n x = 0$

<u>Gaussian Integral.</u>

Consider simplest example $\mathbb{R}$, $\Omega = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$ volume Gaussian.

integration map (polynomial functions)

$$\int : \mathbb{R}[x] \longrightarrow \mathbb{C}$$
$$g(x) \longmapsto \int_{\mathbb{R}} g(x)\, \Omega$$

$$\int_{\mathbb{R}} \left(e^{\left(\underbrace{-\frac{1}{2}x^2}_{\text{free}} + \underbrace{\frac{\lambda}{3!}x^3}_{\text{interaction}}\right)/\hbar} \right) \frac{dx}{\sqrt{\hbar}}$$

more generally

$$\left(\int : \mathbb{R}[x, \theta] \longrightarrow \mathbb{C} \quad g(x) + h(x)\theta \longmapsto \int_{\mathbb{R}} g(x)\, \Omega \right)$$

BV-operator $\Delta = \partial/\partial x\, \partial/\partial\theta - x\,\partial/\partial\theta$

Consider $\forall g(x) \in \mathbb{R}[x] \Rightarrow \Delta g = 0$ since no θ in $g(x)$

Notation $[g]_\Delta$ as Δ-coh class of g for $g_1, g_2 \in \mathbb{R}[x]$

($[g_1]_\Delta = [g_2]_\Delta \Leftrightarrow g_1 - g_2 = \Delta\eta$, $^\exists\eta \in \mathbb{R}[x,\theta]$)

$\Rightarrow \int$ well-defined with Δ-coh class.

$$\int g_1 \Omega = \int g_2 \Omega \Leftarrow [g_1]_\Delta = [g_2]_\Delta$$

Normalization : $\int 1\,\Omega = 1$ proved by gaussian integral.

Example. consider $\eta = x^{m-1}\theta \in \mathbb{R}[x,\theta]$

$$\Rightarrow \Delta(x^{m-1}\theta) = (\partial/\partial x\, \partial/\partial\theta - x\partial/\partial\theta)(x^{m-1}\theta)$$

$$= (m-1)x^{m-2} - x^m$$

$$\Rightarrow [(m-1)x^{m-2}]_\Delta = [x^m]_\Delta$$

$$\Rightarrow (m-1)[x^{m-2}]_\Delta = [x^m] \rightarrow \text{if } m \text{ odd } [x^m] = 0$$

$$\Rightarrow (2k-2)!!\,[1]_\Delta = [x^{2k}]$$

$$\Rightarrow \int_{\mathbb{R}} x^{2k}\,\Omega = (2k-1)!! \int_{\mathbb{R}} 1\cdot\Omega = (2k-1)!!$$

Organize the structure of integral

$\Rightarrow$ Consider following operator $u = e^{\frac{1}{2}\partial/\partial x\,\partial/\partial x}$

$$\mathbb{R}[x,\theta] \longmapsto \mathbb{R}[x,\theta]$$

Explicitly $u(g(x) + h(x)\theta) = ug(x) + (uh(x))\theta$

$$u = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\,\partial/\partial x\,\partial/\partial x\right)^k \rightarrow \text{well-defined.}$$

$\Rightarrow$ lemma (No prove) $\Delta = u^{-1}(-x\,\partial/\partial\theta)\,u$.

(BV-operator Δ conjugate to $(-x\,\partial/\partial\theta)$ via u.)

We have a cochain isomorphism of complex

$$\Delta = u^{-1}(x\partial_\theta)u.$$

$$u : (\mathbb{R}[x,\theta], \Delta) \xrightarrow{\sim} (\mathbb{R}[x,\theta], x\partial/\partial\theta)$$

$$1 \longmapsto 1$$

if we love this, that's to say

$$\boxed{u\circ(\Delta(\varphi)) = (x\partial/\partial\theta)\,u(\varphi)} = u((u^{-1}(x\partial/\partial\theta)u)(\varphi))$$

Observation: $H^*(\mathbb{R}[x,\theta], -x\partial/\partial\theta) = H^0 = \mathbb{R}.$

Let $[\ ]_{-x\partial/\partial\theta} = [\ ]_{-x\partial_\theta}$ rep $(-x\partial/\partial\theta)$-coh class

Then for any $\left(x^m = (-x\partial/\partial\theta)(-x^{m-1}\theta) \quad m>0\right)$

$[g_1]_{-x\partial_\theta} = [g_2]_{-x\partial_\theta} \qquad g_1, g_2 \in \mathbb{R}[x]$

$\Leftrightarrow \exists \eta \in \mathbb{R}[x,\theta] . \quad (-x\partial/\partial\theta)\eta = g_1 - g_2$

$m=1 \qquad g_1 = x,\ g_2 = 0.$

$h(x) \in \mathbb{R}[x]$ polynomial function.

$\Rightarrow [h(x)]_{x\partial\theta} = [h(0)]_{-x\partial\theta}$

Consider $\forall\ g(x) \in \mathbb{R}[x]$

$$\Delta\text{-coh}\quad [g(x)]_\Delta \xrightarrow[\sim]{u} [u(g(x))]_{-x\partial\theta}.$$

$$\| \qquad\qquad\qquad \|$$

$$u(g)(0)[1]_\Delta \xrightarrow{u} [u(g)(0)]_{x\partial\theta}$$

$\Rightarrow [g(x)]_\Delta = [u(g)(0)]_{-x\partial\theta} \qquad u(g)(0) = M$

In other words:

$$\left(\int_{\mathbb{R}} g(x)\,\Omega = \overbrace{u\,g(x)\big|_{x=0}}^{M} \qquad \cancel{\int_{\mathbb{R}} 1\,\Omega} \;\; 1\right.$$

$$= u\,g(x)\big|_{x=0}$$

$$\left.= e^{1/2\,\partial_x^2}\,g(x)\big|_{x=0}\right) \qquad \forall g \in \mathbb{R}[x]$$

$$\Rightarrow \underline{\text{Prop}} \quad \left(\int_{\mathbb{R}} g(x+a)\,\Omega = e^{1/2\,\partial_a^2}\,g(a) \quad a \sim \right)$$

background field.

$$\int(\ )\,\Omega \Rightarrow u(\ )\big|_{x=\dots}$$

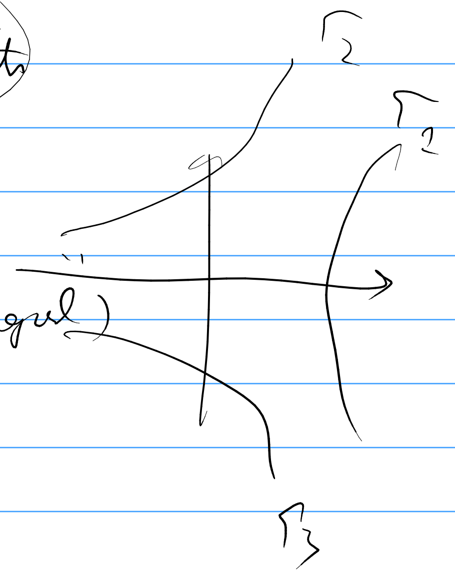
Consider Toy model.

$$I = \int_{\mathbb{R}} e^{(-\frac{1}{2}x^2 + \frac{\lambda}{3!}x^3)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

free $\qquad$ interaction.

$x \to \infty$ $\quad I$ divergent.

① $\mathbb{R} \xrightarrow{\text{deform}} \mathbb{C}$ $\qquad \int_{\Gamma_1}$ ("Airy integral")



change the circle of integration

② Consider asymptotic expansion w.r.t. λ

$$e^{\frac{\lambda}{3!}x^3/\hbar} = \sum_{n \geq 0} \frac{\left(\frac{\lambda}{3!}x^3/\hbar\right)^n}{n!}$$

Consider

② → Perturbation theory.

add background parameter

$$I' = \int_{\mathbb{R}} e^{(-\frac{1}{2}x^2 + \frac{\lambda}{3!}(x+a)^3)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

$$= \sum_{n \geq 0} \int_{\mathbb{R}} e^{-\frac{1}{2\hbar}x^2} \frac{\left(\frac{\lambda}{3!\hbar}(x+a)^3\right)^n}{n!} \frac{dx}{\sqrt{2\pi\hbar}}$$

$g(x+a)$

$$= \sum_{n \geq 0} \int_{\mathbb{R}} \boxed{\frac{\left(\frac{\lambda}{3!\hbar}(x+a)^3\right)^n}{n!}} \left(e^{-\frac{1}{2}\hbar x^2} \frac{dx}{\sqrt{2\pi\hbar}}\right) \sim \Omega$$

$\Rightarrow \int_{\mathbb{R}} (\quad) \Omega$ is described by $e^{\frac{\hbar}{2}\partial_x^2}$

$$I' = e^{\frac{\hbar}{2}\partial_a^2} \cdot e^{\frac{\lambda}{3!}(x+a)^3/\hbar} \Big|_{x=0}$$

$$= e^{\frac{\hbar}{2}\partial_a^2} \cdot e^{\frac{\lambda}{3!}a^3/\hbar}$$

$$= \sum_{k,m \geq 0} \frac{\left(\frac{\hbar}{2}\partial_a^2\right)^k}{k!} \cdot \frac{\left(\frac{\lambda}{3!\hbar}a^3\right)^m}{m!}$$

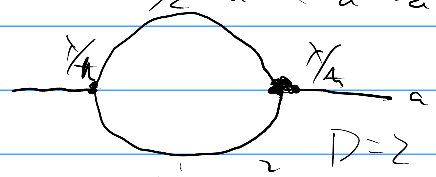
$$\int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2 + \frac{\lambda}{3!}(x+a)^3\right)/\hbar} \, dx/\sqrt{2\pi\hbar}$$

$$= \left(\sum_{k,m\geq 0} \frac{\left(\frac{\hbar}{2}\partial_a^2\right)^k}{k!} \frac{\left(\frac{\lambda}{3!\hbar}a^3\right)^m}{m!} \right) \cdot$$

We want to rep this as a graph.

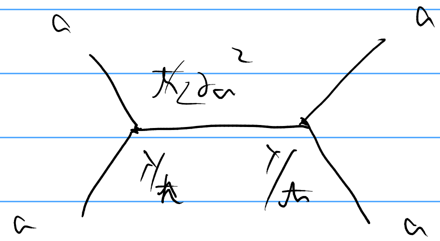
- propagator $\Rightarrow$ rep $\frac{\hbar}{2}\partial_a^2 \leftarrow \hbar$ — gluers. (∂_a act on a.) $\quad \frac{\hbar}{2}\partial_a^2 \quad (\partial_a \otimes \partial_a)$
- cubic vertex $\Rightarrow$ rep $\frac{1}{3!\hbar}a^3 \leftarrow \hbar^{-1}$

Example $\left(\frac{\hbar}{2}\partial_a^2\right)^2 \left(\frac{\lambda}{3!\hbar}a^3\right)^2 \sim a^2$



$D = 2$
$E = 2$
$V = 2$

$$\left(\frac{\hbar}{2}\partial_a^2\right)\left(\frac{\lambda}{3!\hbar}a^3\right)^2$$



$\sim \frac{1}{\hbar}a^4$
$D = 4$
$E = 1$
$V = 2$

In general given a graph Γ let

D = number of external edges
E = number of internal edges
V = number of vertices

Define graph polynomial

$$W_\Gamma(a) = a^D \lambda^V \hbar^{E-V}$$

If Γ connected, $\chi(\Gamma) = V - E = 1 - l$
Euler characteristic
l = number of loops.

$$\Rightarrow \boxed{W_\Gamma(a) = a^D \lambda^V \hbar^{l-1}}$$

$$\Rightarrow \int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2 + \frac{\lambda}{3!}(x+a)^3\right)/\hbar} \, dx/\sqrt{2\pi\hbar}$$

$$:= e^{\frac{\hbar}{2}\partial_a^2} e^{\lambda a^3/3!\hbar} = \exp\left(\sum_{\Gamma \text{ connected trivalent graph}} W_\Gamma(a) / |\mathrm{Aut}(\Gamma)| \right)$$

Aut(Γ) = automorphism group of Γ

↑

(permutations of vertices, internal & external edges)

Formally, we say

$$e^{W(a)/\hbar} = \int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2 + \frac{1}{3!}(x+a)^3\right)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

$$\Rightarrow W(a) := \hbar \sum_{\Gamma:\text{connected}} \frac{W_\Gamma(a)}{|\mathrm{Aut}(\Gamma)|}$$

$$\underset{\hbar}{\overset{\text{expand}}{=}} \sum_{g\geq 0} W_g(a)\,\hbar^g .$$

Here $\hbar W_\Gamma(a) = \hbar^l a^D \lambda^V \sim \hbar^l$ (then ... Ω

we have

$$e^{W(a)/\hbar} = \int_{\mathbb{R}} e^{-\frac{1}{2}x^2/\hbar}\, e^{I(x+a)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

$I(x) = \lambda x^3/3!$ (interaction term).

$$\Leftrightarrow \boxed{e^{W(a)/\hbar} = e^{\frac{\hbar}{2}\partial_a^2}\, e^{I(a)/\hbar}}$$

↑

$(\;)|_{x=a}$ → describe integral.

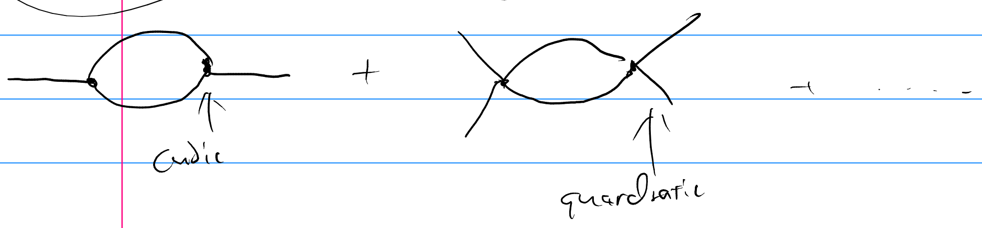
Define $P := \frac{1}{2}\partial_x^2$ propagator.

Defines a transformation on $I(x)$

$I \longrightarrow W(P, I)$,

$e^{W(P,I)/\hbar} \longrightarrow e^{\hbar P} e^{I/\hbar}$ → $\boxed{W(P,I) = \hbar \sum_{\Gamma:\text{connected}} \frac{W_\Gamma}{|\mathrm{Aut}(\Gamma)|}}$

$e^{W(P,I)/\hbar} := e^{\hbar P} e^{I/\hbar}$



$W(P, I)$ w.r.t I.

Proposition: $W(P, -) : \mathbb{R}[[x, \hbar]]^{+} \longrightarrow \mathbb{R}[[x, \hbar]]^{+}$

$\mathbb{R}[[x, \hbar]]^{+} \subseteq \mathbb{R}[[x, \hbar]]$

$$\mathbb{R}[[x, \hbar]]^{+} = x^3 \mathbb{R}[[x]] \oplus \hbar \mathbb{R}[[x, \hbar]]$$

Proof: consider $I \in \mathbb{R}[[x, \hbar]]^{+}$, we need to prove only finitely many graphs contributing when given fixed number external edges & loops.

Consider Γ $\left(\begin{array}{l} l = G \\ D \text{ external edges.} \\ V \text{ vertices} \\ E \text{ internal edges.} \end{array} \right)$

Assume Γ contains $n_{g,k}$ vertices of valency k & loop number g

$$\Rightarrow \quad G = E - V + 1 + \sum_{g,k} g\, n_{g,k}$$

$$V = \sum_{g,k} n_{g,k} \quad , \quad \sum_{g,k} k\, n_{g,k} = 2E + D$$

$$\Rightarrow (2G + D - 2) = \sum_{g,k} \underbrace{(2g + k - 2)}_{\geq 0} n_{g,k}$$

$2g + k - 2 = 0$ iff $g = 1, k = 0$

$\Rightarrow$ only finitely $n_{g,k} \neq 0$. $\Rightarrow 2G + D - 2 \geq 0$
$= 0$ iff $G = 1, D = 0$.

$W(P, -)$ defined as renormalization group flow w.r.t. P.