§ Introduction to QFT : PART I.

§ 1. Interpretation & BV Formalism.

If we consider a physics system.

$$\underbrace{S}_{\text{action functional}} : \underbrace{\mathcal{E}}_{\text{space of fields}} \to \mathbb{C} \qquad (\mathcal{E} \to \mathbb{R})$$

① How do we choose $\mathcal{E}$ fields

$\mathcal{E} : M \to N$. $\to$ M ~~mat~~ mflds $\to$ Riemann, $\to$ smooth metrics, $\to$ dim d.

Boundaries, without boundary

$\to \partial M = \bigcup_i B_i$

$\Rightarrow$ each $B_i \to$ sheaf'ize to $\mathcal{H}_i$ Hilbert space.

path integral $\bigotimes_i \mathcal{H}_i \to \mathbb{C}$

$\mathcal{H}_i \to$ unitary operator group $U_i(t) = e^{-i/\hbar \hat{H}_i t}$

$S : \mathcal{E} \to \mathbb{C}$

In classical physics

critical locus $\mathrm{Crit}(S) = \{ \delta S = 0 \}$ $\leftarrow$ asymptotically $\hbar \to 0$

Quantum case
$$\int_{\mathcal{E}} \underbrace{\mathcal{O}}_{\text{observable}} e^{i/\hbar S}$$

$\dim(\mathcal{E}) \to \infty$ Path Integral.

why not rigoriously defined

① Lebesgue type flat measure $\mathcal{D}\gamma$ / cannot be defined

② (1960. Cameron) $\left(\exp\left(\frac{1}{\hbar} \int_0^t \frac{m}{2} \dot{r}(s)^2 ds \, \mathcal{D}_\gamma \right) \Big/ \int (\sim) \right)$
Feynman measure, Real.

finite dim approximation $\to$ resulting measure $\to$ infinite measure D

$(\hbar \to 0)$

$S : \mathcal{E} \to \mathbb{C}$

① $\mathcal{E} = C^\infty(X)$ → dim = 0 QFT → Scalar FT.

$S[\phi] = \int_X |d\phi|^2 + m^2\phi^2$, $\phi \in C^\infty(X)$

consider V vector bundle

② $\mathcal{E} = \{$ connection on $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ $\}$ → gauge theory

$YM[A] = \int \mathrm{Tr}\, F \wedge * F$, $F = dA + \frac{1}{2}[A, A]$

$CS[A] = \frac{1}{2}\int \mathrm{Tr}\, A \wedge dA + \frac{1}{6}\int \mathrm{Tr}\, A \wedge [A, A]$

($\dim X = 3$)

③ $\mathcal{E} = \mathrm{map}\{\Sigma \to X\}$ → σ-model.

④ $\mathcal{E} = \{$ metrics on X $\}$ → gravity . . .

$p \to \bigcirc$

Γ $X \to E$

<u>Observables</u>

Suppose consider a QFT on X.

X : space time $\boxed{\mathcal{E} = \Gamma(X, E)}$ fields.

$\dim(X) = d$

we want to consider path integral.

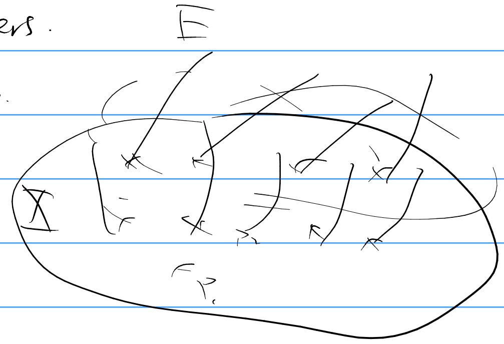
① $\dim(X) = 0$ 0-dim QFT → $X = \{pt\}$

$\Rightarrow$ $\mathcal{E} = \mathbb{R}^n$ ⇝ calculus

distance → metric → top

② $\dim(X) > 0$ → topology on X matters.

$\mathcal{E} \neq \prod_{p \in X} E_p$ $X \in$ top.



new structure ⟹ Observable Algebra.

Roughly speaking

Observable = functions on fields

= $\mathcal{O}(\mathcal{E})$

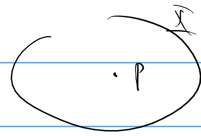
Eg : <u>linear observable = distributions</u>

New structure come from the following fact:

given $U \subset X$

$Obs(U)$ = observables supported in U.

Example: $\boxed{\mathcal{E} = C^\infty(X)}$ $p \in X$, consider.



$O_1 : \mathcal{E} \mapsto \mathbb{R}$

$O_1(f) = f(p)^m$, $m \in \mathbb{Z}_+$, $\forall f \in \mathcal{E}$

O_1 is an observable supported in any nbhd of p.

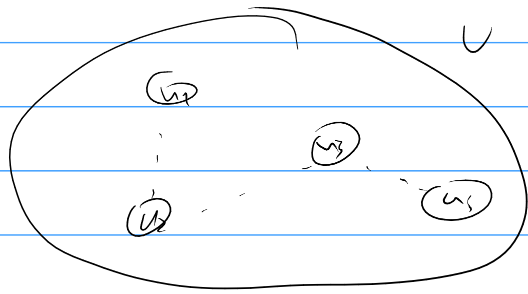
Consider $U \subset X$

Let $\mathcal{E}(U) = \Gamma(U, E)$,

$\Rightarrow$ $obs(U)$ = function on $\mathcal{E}(U)$

$\rightarrow$ New structure. consider $V \subset X$, disjoint open $U_i \subset U$

$\coprod U_i \subset V$



we can define a map.

$\bigotimes_i obs(U_i) \longmapsto obs(V)$

(OPE) : operator product expansion.

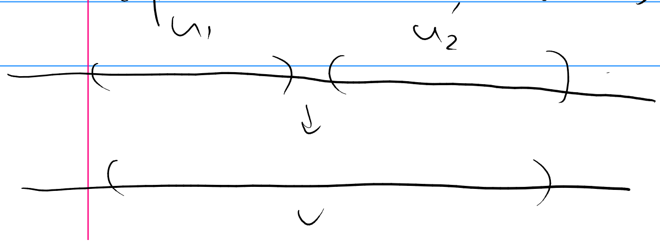
Intuitively $\mathcal{E}(U) \xrightarrow{\text{restriction}} \mathcal{E}(U_i)$

$\Rightarrow$ $\mathcal{O}(\mathcal{E}(U_i)) \longmapsto \mathcal{O}(\mathcal{E}(V))$

$\rightarrow$ $\bigotimes_i obs(U_i) \longmapsto obs(V)$

Example $\dim X = 1$ (QM $\rightarrow$ TQM)

In top QFT, $obs(U)$ only depends on the topology on U.



Let $obs(U) = A$ for U contractible

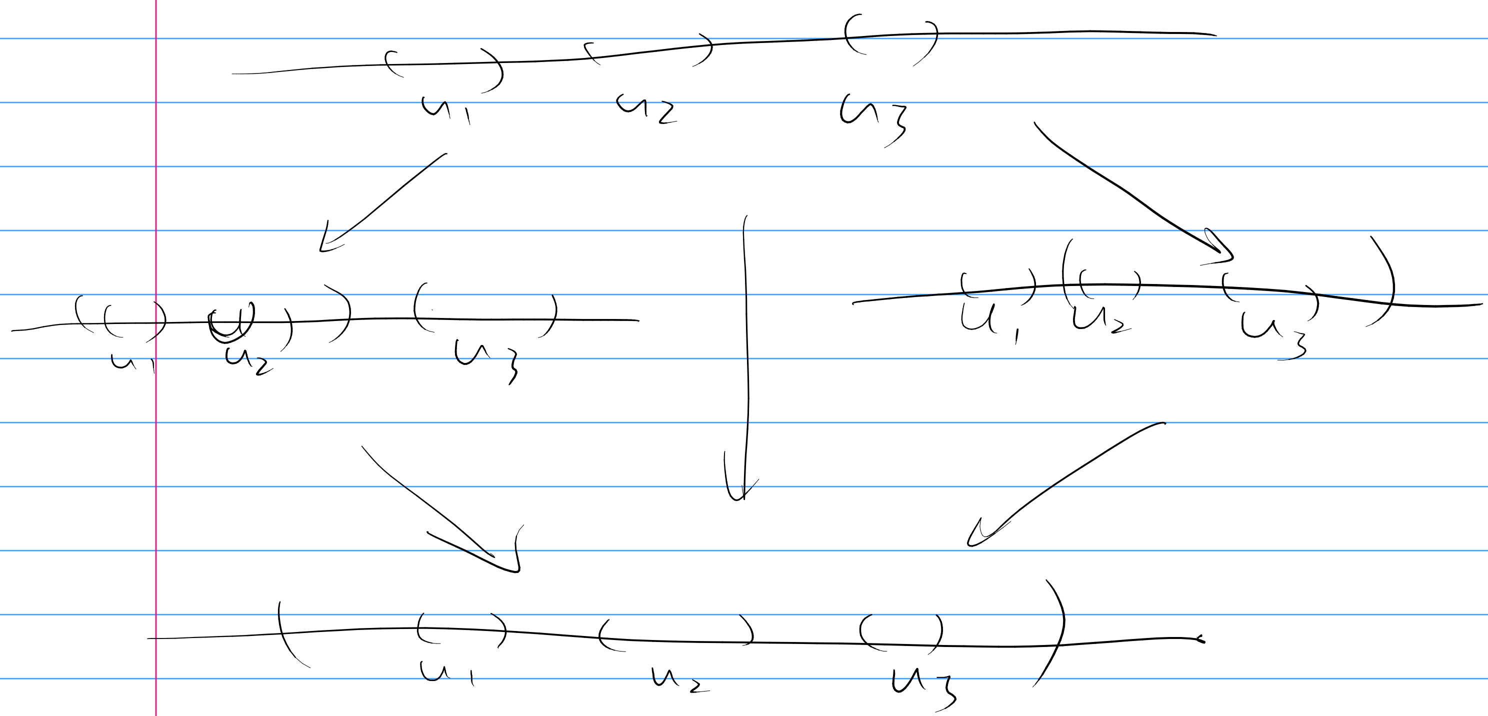
$\mathrm{obs}(U) = A$ for U contractible

$$\Rightarrow \mathrm{obs}(U_1) \otimes \mathrm{obs}(U_2) (\otimes \cdots) \longmapsto \mathrm{obs}(V)$$
$$\quad \overset{\shortparallel}{A} \otimes \overset{\shortparallel}{A} \longmapsto \overset{\shortparallel}{A}$$

→ we can prove this one associative.

Algebraic structure : $H_0(\mathbb{R} - \{0\}) = H_0(\mathbb{R} - \{0\})$
$$= \mathbb{Z}_{left} \oplus \mathbb{Z}_{right}$$

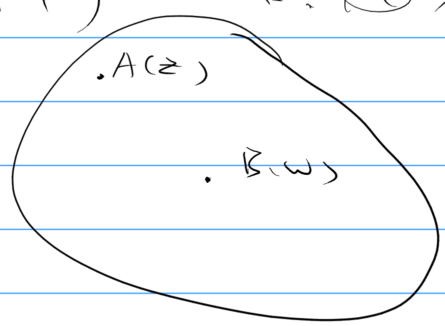
⇒ defines left / right multiplication..



$$\rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c) = (a \cdot b \cdot c)$$

Example 2 $\dim \Sigma = 2$ (Chiral QFT) $\Sigma : U \to \mathbb{C}$

$$A(z) B(w) = \sum_{m \in \mathbb{Z}} \frac{(A_{(m)} B)(w)}{(z-w)^{m+1}}$$



$\{A_{(m)} B\} \to \dim = \infty$

observable algebra ⇒ vertex algebra.

§ BV formalism & Homological integration

$\left(\int = \text{Homology} \right)$

Calculus Revisit

Let M oriented compact manifold, $\dim M = n$

$(\Omega^\bullet(M), d)$ de Rham Complex.

$\left(\int_M : \Omega^\bullet(M) \longmapsto \mathbb{R} \right)$ $\Rightarrow H^n_{dR}(M) = \mathbb{R}$

$\alpha \in \Omega^n(M) \longmapsto \int_M \alpha$

$\Rightarrow$ $\boxed{\left(\int_M \right) = \left(H^n_{dR} \right) \quad \big\downarrow n \to \infty \quad ? \text{ not so sure.}}$

$\int_M : \Omega^n(M) \to H^n_{dR}(M) \simeq \mathbb{R}$

$\alpha \longmapsto [\alpha]$

Batalin-Vilkovisky (BV) formalism.

Refine polyvector field.

$$PV^k(M) := \Gamma(M, \wedge^k TM)$$

$$PV^\bullet(M) = \bigoplus_k PV^k(M)$$

Let Ω be a fixed volume form on M,

$$PV^k(M) \simeq \Omega^{n-k}(M)$$

$$\mu \longleftrightarrow \mu \lrcorner \Omega$$

locally if $\Omega = e^\varphi dx^1 \wedge \cdots \wedge dx^n$

$\mu = \mu^{i_1 \cdots i_k} \partial_{i_1} \wedge \cdots \wedge \partial_{i_k}$

then $\mu \lrcorner \Omega = \sum \pm \mu^{i_1 \cdots i_k} e^\varphi dx^1 \wedge \cdots \wedge \widehat{dx^{i_1}} \wedge \cdots \wedge \widehat{dx^{i_k}} \cdots dx^n$?

$\Delta : PV^k \to PV^{k-1}$ w.r.t Ω

(BV-operator)

$$\begin{array}{ccccccccc}
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n & \xrightarrow{\int} & \mathbb{R} \\
\wr\wr & & \wr\wr & & & & \wr\wr & \nearrow_{\int_{BV}} & \\
PV^n & \xrightarrow{\Delta} & PV^{n-1} & \xrightarrow{\Delta} & \cdots & \xrightarrow{\Delta} & PV^0 & &
\end{array}$$

H^n_{dR}

Eg: $\partial : PV^1 = Vect(M) \longrightarrow PV^0 = C^\infty(M)$

$\int_{BV} : PV^0 \longrightarrow \mathbb{R}$, $f \longmapsto \int f \Omega$

Homologically: $\int_{BV} = H^0_{BV} \leftarrow n < \infty$

($n \to \infty$ $\to$ renormalization) $\Rightarrow$ Homological integration)

<u>Explicit rep of Δ</u>

locally in $U \subset M$, let $\{x_i\}$ local coords.
$\dim M = n$.

$$\Omega = e^{f(x)} dx^1 \wedge \cdots \wedge dx^n$$

$$PV(U) = C^\infty(U) \cdot [\partial_1, \ldots, \partial_n], \qquad \partial_i \partial_j + \partial_j \partial_i = 0$$

let $\theta_i = \partial_i$, $\mu \in PV(X)$ locally $\Rightarrow$

$$\mu = \mu(x_i, \theta_i), \qquad \{\theta_i, \theta_j\} = 0$$

let $\partial/\partial\theta_i$ be derivative w.r.t. θ_i

$$\Rightarrow \left(\Delta = \sum_i \partial/\partial x_i \, \partial/\partial\theta_i + \sum_i (\partial_i f) \, \partial/\partial\theta_i \right)$$

2nd order 1st operator