Algebraic Surfaces 2022-23

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1. Introduction

This will be an introduction to the Enriques classification of (complex) algebraic surfaces. We consider smooth projective surfaces over an algebraically closed field and the classification in question is the classifcation up to birational equivalence.

For curves, there is a (up to isomorphism) unique smooth projective model for each birational class and the main numerical invariant is the (geometric) genus. For surfaces the smooth projective model is not unique, however, there is a way to classify using a relative minimal model. In this case the main invariant is the Kodaira dimension κ of a surface, which can assume one of the four values $-\infty, 0, 1$ and 2. We will see that the surfaces with $\kappa = 2$ are of 'general type' and give a more explicit classification for surfaces with Kodaira dimension $\kappa < 2$.

Our main references are:

References

[1] Beauville, Arnaud: *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series, 68, Cambridge University Press, Cambridge, 1983.

[2] Hartshorne, Robin: *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

2. Preliminaries

We summarize some of the results required in the sequel. In what follows X will be a smooth projective variety over an algebraically closed field k. In particular, by a result of Serre [2, III, Corollary 7.7] for every coherent sheaf (or coherent $\mathcal{O}_X$-module) $\mathcal{F}$ on X and every $i \geq 0$ the cohomology groups $\mathrm{H}^i(X, \mathcal{F})$ are finite dimensional k-vector spaces.

A Weil divisor on X is an element of the free abelian group $\mathrm{Div}(X)$ generated by the closed subvarieties of codimension 1, i.e. has the form

$$D = \sum_{i=1}^{r} n_i Y_i,$$

where $n_i \in \mathbb{Z}$ and $Y_i \subseteq X$ is a closed subvariety of codimension 1. Such a D is efffective, if $n_i \geq 0$ for all i. Given a function $f \in k(X)^\times$, we can associated to f a Weil divisor $\mathrm{div}(f)$ (the sum of zeros minus the sum of poles). A Weil divisor is principal, if it is the divisor of a function and two Weil divisors D, D' are linearly equivalent, $D \sim D'$, if $D - D' = \mathrm{div}(f)$ for some $f \in k(X)^\times$. We write $\mathrm{Cl}(X)$ for the quotient $\mathrm{Div}(X)/\sim$. Since X is projective, the kernel of the map $f \mapsto \mathrm{div}(f)$ ist just $k^\times$ and the above discussion is summarized by the following exact sequence

$$0 \to k^\times \to k(X)^\times \overset{\mathrm{div}}{\to} \mathrm{Div}(X) \to \mathrm{Cl}(X) \to 0.$$

The degree map $\deg : \mathrm{Div}(X) \to \mathbb{Z}$, $\sum n_i Y_i \mapsto \sum n_i$ descends to a map $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$, in particular linearly equivalent Weil divisors have the same degree.

A Cartier divisor on X is an element of $\mathrm{CaDiv}(X) = \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of X. Every Cartier D divisor can be represented by an open covering $\{U_i\}$ of X and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}_X^\times)$ such that for i, j we have $f_i \cdot f_j^{-1} \in \Gamma(U_i \cap U_j), \mathcal{O}_X^\times)$; set $D = \{U_i, f_i\}$. A Cartier divisor $D = \{U_i, f_i\}$ is effective, if all $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$; it is principal, if it is in the image of the map $\Gamma(X, \mathcal{K}_X^\times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. Two Cartierdivisors D, D' are linearly equivalent, $D \sim D'$, if there difference is principal; set $\mathrm{CaCl}(X) = \mathrm{CaDiv}(X)/\sim$.

There is a map [2, II, Proposition 6.8]

$$\psi : \mathrm{CaDiv}(X) \to \mathrm{Div}(X),\ D = \{U_i, f_i\} \mapsto \sum v_Y(f_i)Y,$$

where the sum runs over all closed subvarieties $Y \subseteq X$ of codimension 1, and the coefficients come from any index i such that $Y \cap U_i \neq \emptyset$. This is well-defined and under our assumptions on X an isomorphism. Under this isomorphism effective Cartier divisors correspond to effective Weil divisors and principal Cartier divisors correspond to principal Weil divisors. In particular, the map ψ induces an isomorphism

$$\overline{\psi} : \mathrm{CaCl}(X) \overset{\cong}{\to} \mathrm{Cl}(X).$$

A line bundle (or invertible sheaf) $\mathcal{L}$ on X is a locally free sheaf (or locally free $\mathcal{O}_X$-module) of rank 1. The isomorphism classes of line bundles on X form the Picard group $\mathrm{Pic}(X)$; this is an abelian group with respect to $|\mathcal{L}| + \mathcal{L}'| = |\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'|$. Given a Cartier divisor $D = \{U_i, f_i\}$, we can associate to D a subsheaf $\mathcal{L}(D) \subseteq \mathcal{K}_X$, where $\mathcal{L}(D)$ is the sub $\mathcal{O}_X$-module generated by f_i^{-1} on U_i. This is a line bundle and as a special case of [2, II, Proposition 6.13] the map

$$\Theta : \mathrm{CaCl}(X) \to \mathrm{Pic}(X),\ |D| \mapsto |\mathcal{L}(D)|$$

is an isomorphism of abelian groups; in particular, $\mathcal{L}(0) = \mathcal{O}_X$.

In summary, we have for every smooth projective variety X isomorphisms

$$\mathrm{Pic}(X) \overset{\cong}{\leftarrow} \mathrm{CaCl}(X) \overset{\cong}{\to} \mathrm{Cl}(X).$$

Line bundles on X give rise to morphisms $X \to \mathbb{P}_k^n$ as follows. Assume $f : X \to \mathbb{P}_k^n$ is a morphism. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. The homogenous coordinates $X_0, \dots, X_n$ give rise to global sections $X_0, \dots, X_n$ of $\mathcal{O}_{\mathbb{P}^n}(1))$ which generate $\mathcal{O}_{\mathbb{P}^n}(1)$ (i.e. for every point P the images of these sections generate the stalk $\mathcal{O}_{\mathbb{P}^n}(1)_P$ of the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ as a module over the local ring $\mathcal{O}_P$). The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ pulls back to a line bundle $f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ on X which is generated by the global sections $s_i = f^*(X_i)$, $i = 0, \dots, n$. It is a theorem [2, II, Theorem 7.1] that the converse holds: Given a line bundle $\mathcal{L}$ on X and global sections $s_0, \dots, s_n$ which generate $\mathcal{L}$, there is a unique morphism $f : X \to \mathbb{P}_k^n$ such that $\mathcal{L} \cong f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and $s_i = f^*(X_i)$, i.e.

$$\left\{ \begin{array}{l} \mathcal{L} \text{ line bundle on } X \text{ generated by global} \\ \text{sections } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \end{array} \right\} \rightsquigarrow \{f : X \to \mathbb{P}_k^n \text{ morphism}\}$$

The above correspondence can be reformulated as follows: Let $\mathcal{L}$ be a line bundle on X and let $0 \neq s \in \Gamma(X, \mathcal{L})$. If $U \subseteq X$ is such that $\phi : \mathcal{L}|_U \xrightarrow{\cong} \mathcal{O}_U$, then $\phi(s) \in \Gamma(U, \mathcal{O}_X)$. This, applied to an open covering where $\mathcal{L}$ is locally trivial gives rise to an effective Cartier divisor (the divisor of zeros of s) $\operatorname{div}(s)_0 = \{U_i, \phi_i(s)\}$. If D_0 is any divisor on X and $\mathcal{L} = \mathcal{L}(D_0)$ is the associated line bundle, the divisor $\operatorname{div}(s)_0$ is linearly equivalent to D_0 and $s \mapsto \operatorname{div}(s)_0$ defines a map

$$\eta : \Gamma(X, \mathcal{L}) \setminus \{0\} \to |D_0| = \{D \mid D \text{ effective divisor}, D \sim D_0\}.$$

It can be shown [2, II, Proposition 7.7] that this map is surjective with kernel isomorphic to $k^\times$, i.e. η induces an isomorphism

$$\Gamma(X, \mathcal{L}) \setminus \{0\})/k^\times \xrightarrow{\cong} |D_0|.$$

The set $|D_0|$ (possible empty) is a complete linear system on X. The above isomorphism shows that if $\Gamma(X, \mathcal{L}) \cong k^m$, the set D_0 is isomorphic to $\mathbb{P}_k^{m-1}$. A linear system on X is a subset $\mathfrak{d} \subseteq |D_0|$ of a complete linear system on X which under the above isomorphism correspondes to a linear subspace of the corresponding projective space. Any such $\mathfrak{d}$ corresponds to a sub vector space $V \subseteq \Gamma(X, \mathcal{L})$ (namely $V = \{s \in \Gamma(X, \mathcal{L}) \mid \operatorname{div}(s)_0 \in \mathfrak{d}\} \cup \{0\}$); the dimension of $\mathfrak{d}$ is the dimension as a projective space, i.e. $\dim \mathfrak{d} = \dim V - 1$. A point $P \in \mathfrak{d}$ is a base point, if $P \in \operatorname{Supp}(D)$ for all $D \in \mathfrak{d}$ (here $\operatorname{Supp}(D)$ is the union of the prime divisors of D. If $\mathfrak{d}$ corresponds to a subspace $V \subseteq \Gamma(X, \mathcal{L})$, then $\mathfrak{d}$ is base point free if and only if $\mathcal{L}$ is generated by global sections in V. Thus

$$\left\{ \begin{array}{l} \mathfrak{d} \text{ linear system on } X \text{ without base} \\ \text{points}; s_0, \ldots, s_n \text{ basis of } V \end{array} \right\} \rightsquigarrow \{f : X \to \mathbb{P}_k^n \text{ morphism}\}$$

For example, let $X = \mathbb{P}_k^n$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}_k^n}(m)$ with $m \geq 0$ and consider the complete linear system $|\mathcal{O}_{\mathbb{P}_k^n}(m)|$. we have: $\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)) \cong S_m$ is a k-vector space of dimension $N = \binom{m+n}{m}$ which is generated by the homogenous polynomials of degree m in the variables $X_0, \ldots, X_n$. Since these polynomials have no common zeros, the complete linear system $|\mathcal{O}_{\mathbb{P}_k^n}(m)|$ is base point free and gives rise to a morphism $\nu : \mathbb{P}_k^n \to \mathbb{P}_k^{N-1}$; this is the Veronese morphism which is a closed embedding whose image $\nu(\mathbb{P}_k^n) \subseteq \mathbb{P}_k^{N-1}$ is the Veronese variety.

We will need a particular line bundle: Since for a locally free sheaf $\mathcal{F}$ on X of rank m the exterior power $\wedge^r \mathcal{F}$ is a locally free sheaf on X of rank $\binom{m}{r}$ [2, II, Exercise 5.16] the exterior power $\det(\mathcal{F}) = \wedge^n \mathcal{F}$ defines a line bundle on X. In particular, since $\Delta : X \to X \times_k X$ is a closed embedding, $\Delta(X) \subseteq X \times_k X$ is a closed subset and defines an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{X \times X}$. The sheaf of Kähler differentials (or cotangent bundle) on X is the pullback $\Omega_X = \Omega_X^1 = \Delta^*(\mathcal{J}/\mathcal{J}^2)$; if X is smooth of dimension n, the sheaf Ω_X is a locally free sheaf of rank n [2, Theorem 8.15]. Set $\Omega_X^p = \wedge^p \Omega_X$. Then $\omega_X = \Omega_X^n$ is a line bundle, this is the canonical bundle on X. For example, if $X = \mathbb{P}^n_k$, then there is an exact sequence of sheaves [2, II, Theorem 8.13]

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0,$$

it follows from the formula $\det(\mathcal{O}_X(-1)^{n+1}) = \det(\Omega_X) \otimes \det(\mathcal{O}_X)$ that $\omega_X = \mathcal{O}_X(-n-1)$.

If X is a smooth projective variety of dimension n with canonical sheaf ω_X and $\mathcal{F}$ is a locally free shreaf, by Serre Duality [2, III, Corollary 7.7]

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee \text{ for all } i \geq 0.$$

For example, if $X = \mathbb{P}^n_k$ and $\mathcal{F} = \mathcal{O}_X(m)$, then $\omega_X \cong \mathcal{O}_X(-n-1)$ and $\mathcal{O}_X(m)^\vee \otimes \mathcal{O}_X(-n-1) \cong \mathcal{O}_X(-m-n-1)$. It follows from Serre Duality that $\mathrm{H}^0(X, \mathcal{O}_X(m)) \cong \mathrm{H}^n(X, \mathcal{O}_X(-m-n-1)^\vee$, cp. [2, III, Theorem 5.1].

For a smooth projective variety X the arithmetic and geometric genus are defined as $p_a(X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X)$ and $p_g(X) = \dim_k \mathrm{H}^0(X, \omega_X)$. Both $p_a(X)$ and $p_g(X)$ are finite non-negative integers. By [2, II, Theorem 8.18] the geometric genus is a birational invariant, i.e. if two smooth projective varieties are birationally equivalent, $X \sim Y$, then $p_g(X) = p_g(Y)$; in particular, if two such varieties have different geometric genera, they cannot be birationally equivalent.

3. Curves

In this section we will use the word 'curve' to refer to a smooth projective curve. For a curve X we have $\omega_X = \Omega^1_X$, thus

$$p_a(X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X) \text{ and } p_g(X) = \dim_k \mathrm{H}^0(X, \omega_X) = \mathrm{H}^0(X, \Omega^1_X).$$

Lemma 3.1. *Let X be a curve. Then $p_a(X) = p_g(X)$; we will refer to this nonegative integer as the genus $g(X) = p_a(X) = p_g(X)$ of the curve X.*

• If X is a curve over $\mathbb{C}$, it's underlying topological space is an orientable compact 2-dimensional $\mathbb{R}$-manifold. Such a topological space is homeomorphic to a sphere with handles, the genus of the curve is the number of handles.

• There are curves of every genus $g \geq 0$: For example, if $Q \subseteq \mathbb{P}^3_k$ is the smooth quadric defined by $xy = uv$, then $\mathrm{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. If D is a divisor on Q, we say D has type (a, b) if $|\mathcal{L}(D)| = (a, b) \in \mathrm{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. For every $a, b > 0$ there is an irreducible smooth curve $D \subseteq Q$ of type (a, b) and for this curve $p_a(D) = ab - a - b - 1$. In particular, the curve D of type $(g + 1, 2)$ has genus $g(D) = g$ [2, III, Exercise 5.6(c)].

• We will see that for a smooth proective hypersurface $X \subseteq \mathbb{P}^2_k$ of degree d the genus is $g(X) = \frac{1}{2}(d-1)(d-1)$. In particular, there are curves, which do not arise as such hypersurfaces.

Proof. By Serre Duality $\mathrm{H}^1(X, \mathcal{O}_X) \cong \mathrm{H}^0(X, \mathcal{O}_X^\vee \otimes \omega_X)^\vee$, where $\mathcal{O}_X^\vee \otimes \omega_X \cong \omega_X$. Hence the k-vector spaces $\mathrm{H}^1(X, \mathcal{O}_X)$ and $\mathrm{H}^0(X, \omega_X)$ are dual to each other and have the same dimension. □

Let D be a disivor on a curve X and consider the corresponding complete linear system $|D|$, which is isomorphic to some projective space. If $\mathcal{L}(D)$ is the line bundle associated to D, let $l(D) = \dim_k \mathrm{H}^0(X, \mathcal{L}(D))$. Then the dimension of $|D|$ is $l(D) - 1$.

Lemma 3.2. *Let D be a divisor on a curve X. If $l(D) \neq 0$, then* $\deg(D) \geq 0$. *If $l(D) \neq 0$ and* $\deg(D) = 0$, *then $D \sim 0$, i.e. $\mathcal{L}(D) = \mathcal{O}_X$.*

Proof. If $l(D) \neq 0$, then $|(D)| \neq \emptyset$ and there is an effective divisor D' with $D \sim D'$. Since the degree of a divisor depends only on the linear equivalence class, then $\deg(D) \geq 0$. If $l(D) \neq 0$ and $\deg(D) = 0$, then D is linearly equivalent to an effective divisor of degree 0. The only such divisor is 0. $\square$

Let K_X be a canonical divisor on X, i.e. a divisor such that $\mathcal{L}(K_X) = \omega_X$.

Theorem 3.3. *(Riemann-Roch for curves) Let X be a curve of genus g, D any divisor on X, and K_X a canonical divisor. Then we have the formula*

$$(\#) \quad l(D) - l(K_X - D) = \deg(D) + 1 - g.$$

Proof. Since $\mathcal{L}(K_X) = \omega_X$, we have $\mathcal{L}(K_X - D) \cong \mathcal{L}(D)^\vee \otimes \omega_X$ and it follows from Serre Duality that $\mathrm{H}^0(X, \mathcal{L}(D)^\vee \otimes \omega_X)$ is dual to $\mathrm{H}^1(X, \mathcal{L}(D)$, i.e. these k-vector spaces have the same dimension.

For a coherent sheaf $\mathcal{F}$ on a curve X consider the Euler characteristic

$$\chi(\mathcal{F}) = \sum_i (-1)^i \dim_k \mathrm{H}^i(X, \mathcal{F}) = \dim_k \mathrm{H}^0(X, \mathcal{F}) - \dim_k \mathrm{H}^1(X, \mathcal{F}).$$

Then

$$l(D) - l(K_X - D) = \dim_k \mathrm{H}^0(X, \mathcal{L}(D) - \dim_k \mathrm{H}^1(X, \mathcal{L}(D)) = \chi(\mathcal{L}(D))$$

and we need to show that $\chi(\mathcal{L}(D)) = \deg(D) + 1 - g$. If $D = 0$ is the zero divisor, then $\deg(D) = 0$, $\mathcal{L}(0) = \mathcal{O}_X$ and since $\dim_k \mathrm{H}^0(X, \mathcal{O}_X) = 1$ (because X is projective) and $\dim_k \mathrm{H}^1(X, \mathcal{O}_X) = g$ (by definition), the above formula holds in this case.

For the general case it suffices to show: If D is a divisor and P is a point on X, then (#) hold for D if and only if (#) holds for $D + P$ (since every divisor D can be reached from 0 by adding and subtracting a finite number of points, this proves then theorem).

Consider $i : Y = \{P\} \to X$ as a closed subscheme. Then $\mathcal{O}_Y$ (more precisely $i_*\mathcal{O}_P$) is the skyscaper sheaf k on X which is supported on Y and $\mathcal{L}(-Y)$ is the corresponding ideal sheaf [2, II, Proposition 6.18]. From tensoring the exact sequence

$$0 \to \mathcal{L}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

with $\mathcal{L}(D + Y)$ one obtains the exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + Y) \to \mathcal{O}_Y \to 0$$

and since the Euler characteristic χ is additiv on exact sequences the formula

$$\chi(\mathcal{L}(D + Y)) = \chi(\mathcal{L}(D)) + 1.$$

Because $\deg(D+Y) = \deg(D)+1$, (#) holds for D if and only if (#) holds for $D+P$. $\square$

Corollary 3.4. *Let X be a curve with genus g and canonical divisor K_X. Then* $\deg(K_X) = 2g-2$

Proof. By definition $l(K_X) = \dim_k \mathrm{H}^0(X, \mathcal{L}(K_X)) = \dim_k \mathrm{H}^0(X, \omega_X) = g$ and $l(0) = \dim_k \mathrm{H}^0(X, \mathcal{O}_X) = 1$. By Riemann-Roch 3.3 (with $D = K_X$) then $g-1 = \deg(K_X)+1-g$, i.e. $\deg(K_X) = 2g-2$. $\square$

Examples 3.5. (a) Assume a curve X is rational, i.e. $X \sim \mathbb{P}^1_k$. Then $X \cong \mathbb{P}^1_k$ (since X is smooth, see for example [2, I, Proposition 6.8]), i.e. a curve X is rational if and only if $X \cong \mathbb{P}^1_k$. The formula for the genus of a hypersurface in $\mathbb{P}^2_k$ shows that $g(\mathbb{P}^1_k) = 0$, hence rational curves have genus 0. Assume conversely X is a curve with genus $g(X) = 0$. Let P and Q be two distinct points on X and set $D = P - Q$. By Corollary 3.4 $\deg(K_X) = -2$, thus $\deg(K_X - D) = \deg(K_X) = -2$. It follows from Lemma 3.2 that $l(K_X - D) = 0$ and from Riemann-Roch 3.3 that $l(D) = 1$. Using again Lemma 3.2 we have, since $\deg(D) = 0$, $D \sim 0$, i.e. $P \sim Q$. By [2, II, 6.10.1] this implies X is rational. In summary: $X \sim \mathbb{P}^1_k \Leftrightarrow X \cong \mathbb{P}^1_k \Leftrightarrow g(X) = 0$.

(b) A curve X is elliptic, if $g(X) = 1$. In this case $\deg(K_X) = 0$ by Corollary 3.4 and $l(K_X) = \dim \mathrm{H}^0(X, \omega_X) = g(X) = 1$, hence $K_X \sim 0$ by Lemma 3.2.

Let X be an elliptic curve and P_0 a point of X. Write $\mathrm{Pic}^0(X)$ for the kernel of the map $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$, $|\mathcal{L}(D)| \mapsto \deg(D)$. Consider the map

$$\psi : X \to \mathrm{Pic}^0(X),\ P \mapsto |\mathcal{L}(P - P_0)|.$$

We claim this is a bijection: It suffices to show that given a divisor D of degree 0 on X, there is a unique point P on X such that $D \sim P - P_0$. From Riemann-Roch 3.3, applied to the divisor $D + P_0$, we have

$$l(D+P_0) - l(K_X - D - P_0) = 1 + 1 - 1 = 1.$$

Because $\deg(K_X) = 0$ we have $\deg(K_X - D - P_0) = -1$ and therefore $l(K_X - D - P_0) = 0$. Hence $l(D+P_0) = 1$ and $\dim |D+P_0| = 0$. This means $|D+P_0| \cong \mathbb{P}^0_k = \{\star\}$, i.e. there is a unique effective divisor on X which is linearly equivalent to $D + P_0$. Since the degree of this divisor is 1, it must be a single point P, hence $D + P_0 \sim P$ and so $D \sim P - P_0$.

Assuem for simplicity that $\mathrm{char}(k) \neq 2, 3$. Then every elliptic curve X is given by an affine Weierstrass equation of the form $X : y^2 = x^3 + ax + b$ with $a, b \in K$. Because X is smooth, the discriminant $\Delta(X) = -16(4a^3 + 27b^2) \neq 0$. The j-invariant of X is defined as $j(X) = -1728 \cdot (4a)^3/\Delta \in k$. By [2, IV, Theorem 4.1] two elliptic curves X and X' are isomorphic if and only if $j(X) = j(X')$ and every element of the field k occurs as the j-invariant of some elliptic curve over k. In particular, the map $X \mapsto j(X)$ is a bijection between the set of isomorphism classes of elliptic curves over k and the elements of the field k.

(c) A curve X of genus $g(X) \geq 2$ is called a curve of general type. There are some results concerning a classification of curves of genus 2 (see, for example, [2, IV, Exercise 2.2]); however, such a classification for general $g \geq 2$ is much more difficult and one cannot give an explicit general answer (cp. [2, pg. 345-347]).

4. Geometry on surfaces and the Riemann-Roch Theorem

In this section a surface will be a smooth projective surface X over an algebraically closed field k. A curve on such a surface will be any effective divisor, in particular, it could be singular, reducible or have multiple components. A smooth curve is a curve $C = \sum n_i Z_i$ whose components Z_i are smooth and pairwise disjoint. A curve C is connected, if $C \neq 0$ and the union of the components Z_i is connected. A point will be a closed point.

We want to consider the internal geometry on such a surface; more precisely, the intersection theory on X. If C, D are smooth curves on X which intersect transversally, their intersection $C.D$ will be the $\#(C \cap D)$, i.e. the number of points in $C \cap D$. We will define a general intersection pairing and prove the Riemann-Roch theorem for surfaces, which correlates the dimension of a linear system with certain intersection numbers on the surface.

We will show first:

Theorem 4.1. *Let X be a surface. There is a unique intersection pairing*

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}, \ (C, D) \mapsto C.D$$

with the following properties:

(1) *If C and D are smooth curves meeting transversally, then $C.D = \#(C \cap D)$, the number of points of $C \cap D$,*
(2) *$C.D = D.C$,*
(3) *$(C_1 + C_2).D = C_1.D + C_2.D$,*
(4) *If $C_1 \sim C_2$, then $C_1.D = C_2.D$.*

We begin with the question of what a transversal intersection should be: This is a local question, let $P \in \mathbb{A}^2_k$ be a point and consider the local ring of $\mathbb{A}^2_k$ at P, i.e. $\mathcal{O}_{\mathbb{A}^2_k,P} = \mathcal{O}_P = \{f/g \mid f, g \in k[x, y], g(P) \neq 0\}$. Then $\mathcal{O}_P^\times = \{f/g \mid f(p) \neq 0 \neq g(P)\}$. There is a surjective evaluation map $\mathcal{O}_P \to k, \ f/g \mapsto f(P)/g(P)$ whose kernel is given by the ideal $I_P = \{f/g \mid f, g \in k[x, y], \ g(P) \neq 0, \ f(P) = 0\}$. If f, g are two distinct (irreducible) polynomials in $k[x, y]$, we have inclusions $k[x, y] \to \mathcal{O}_P \to k(x, y)$ and we can consider the ideal defined by the polynomials f and g in $\mathcal{O}_P$ and the k-vector space $\mathcal{O}_P/(f, g)$. Set $(f.g)_P = \dim_k \mathcal{O}_P/(f, g)$.

If $f(P) \neq 0$, then $f \in \mathcal{O}_P^\times$ and $(f, g) = \mathcal{O}_P$, i.e. $(f.g)_P = 0$. If $f(P) = 0 = g(P)$, then evaluation defines a surjective map $\mathcal{O}_P/(f, g) \to k$ and $(f.g)_P \geq 1$; furthermore, $(f.g)_P = 1$ if and only if $(f, g) = I_P$.

Definition 4.2. Let C, D be two distinct irreducible curves on a surface X, $P \in C \cap D$ and f (resp. g) a local equation of C in $\mathcal{O}_{X.P}$ (resp. D in $\mathcal{O}_{X,P}$). Then the intersection mutiplicity of C and D at the point P is defined as

$$(C.D)_P = (f.g)_P = \dim_k \mathcal{O}_{X,P}/(f, g).$$

The curves C and D intersect transverally at P if $(C.D)_P = 1$.

NB. By the Nullstellensatz $(f.g)_P$ is a non-negative integer.
- $(f.g)_P = (g.f)_P$,
- $(f.g)_P = (f.(g + fh))_P$ for all h (since $(f, g) = (f, g + fh)$),
- $(f.(gh))_P = (f.g)_P + (f.h)_P$ (this follows from the exactness of the sequence

$$0 \to \mathcal{O}_P/(f, h) \to \mathcal{O}_P/(f, gh) \to \mathcal{O}_P/(f, g) \to 0).$$

Examples 4.3. (a) Let $0 \in \mathbb{A}^2_k$ be the origin. Then $I_0 = (x, y) \subseteq k[x, y]$ and for the coordinate axes we have locally $(x.y)_0 = 1$, as expected.
(b) Consider locally the curves $C = V(y)$ and D, where $D = V(f)$ for some irreducible $f \in k[x, y]$ and $C \neq D$. Write $f = f(x, 0) + hy$, then $(y.f)_0 = (y.(f(x, 0)))_0$. Because $C \neq D$ the polynominal $f(x, 0) \in k[x]$ is not the zero polynomial. Thus $f(x, 0) = x^n \cdot g$, where $g \in k[x]$ does not vanish at 0 and n is the multiplicity of 0 in $f(x, 0)$. Hence we have

$$(y.f)_0 = (y.(x^n g))_0 = n \cdot (y.x)_0 + (y.g)_0 = n.$$

We also need some geometric considerations, which follow from Bertini's Theorem, see for example [2, II,81.8] and [2, III,7.9.1]. Recall that given a morphism $f : X \to Y$, a line bundle $\mathcal{L}$ on X is very ample, if there is an immersion $i : X \to \mathbb{P}^n_Y$ such that $\mathcal{L} \cong i^*(\mathcal{O}_{\mathbb{P}^n_Y}(1))$. If $Y = \mathrm{Spec}(k)$, this means that $\mathcal{L}$ admits global sections $s_0, \ldots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_k$ is an immersion. A line bundle $\mathcal{L}$ on X is ample, if for every coherent sheaf $\mathcal{F}$ there is an integer n_0 such that for $n \geq n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^n = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections. It's a special case of a Theorem [2, II, Theorem 7.6] that for a variety X over a field k a line bundle $\mathcal{L}$ is ample if and only if for some $m > 0$ the tensor power $\mathcal{L}^m$ is very ample with respect to $X \to \mathrm{Spec}(k)$.

We have the following 'moving lemma':

Lemma 4.4. *Let X be a surface and D a very ample divisor on X (for example, the intersection of X with a hyperplane in some projective embedding $i : X \to \mathbb{P}^n_k$, i.e. D such that $\mathcal{L}(D) \cong i^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$). If $C_1, \ldots, C_r$ are irreducible curves on X, then almost all curves D' in the complete linear system $|D|$ are irreducible, smooth and meet each of the C_i transversally.*

Proof. See [2, V, Lemma 1.2]. □

If C is a irreducible smooth curve on a surface X and $\mathcal{L}(D)$ is a line bundle on X, write $\mathcal{L}(D) \otimes \mathcal{O}_C = \mathcal{L}(D)|_C$ for the pullback of this line bundle to C. Note that on this smooth curve we have an isomorphism $\mathrm{Pic}(C) \cong \mathrm{Cl}(C)$ and therefore a degree map on $\mathrm{Pic}(C)$; in particular, $\deg(\mathcal{L}(D) \otimes \mathcal{O}_C)$ is a well defined integer.

Lemma 4.5. *Let C be a irreducible smooth curve on a surface X and let D be any curve on X which meets C transverally. Then*

$$\#(C \cap D) = \deg(\mathcal{L}(D) \otimes \mathcal{O}_C).$$

Proof. Consider $D \to X$ as a closed subscheme. From the exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

we obtain by tensoring with $\mathcal{O}_C$ the exact sequence

$$0 \to \mathcal{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C\cap D} \to 0,$$

where $C \cap D = C \times_X D$ is the scheme-theoretic intersection. From the last sequence we see that the line bundle $\mathcal{L}(D) \otimes \mathcal{O}_C$ on C corresponds to the closed subscheme $C \cap D$ on C and, since C and D intersect transversally

$$\#(C \cap D) = \deg(\mathcal{L}(D) \otimes \mathcal{O}_C).$$

□

Proof. (of Theorem 4.1) Fix an ample divisor H on X (since X is a projective surface it has a ample line bundle $\mathcal{L}(D)$ and D is such an ample divisor). Given any two divisors C and D on X, we can choose, by definition of ampleness, an integer $k > 0$ such that the line bundles $\mathcal{L}(C + kH), \mathcal{L}(D + kH), \mathcal{L}(kH)$ are all generated by global sections. Then choose $l > 0$ such that $\mathcal{L}(lH)$ is very ample [2, II, Theorem 7.6] and let $n = k + l$. Then $C + nH$, D+nH, nH are all very ample [2, II, Exercise 7.5]. By Lemma 4.4 there are $C' \in |C + nH|$ irreducible smooth, $D' \in |D + nH|$ irreducible smooth and transversal to C', $E' \in |nH|$ irreducibel smooth and transversal to D', and $F' \in |nH|$ irreducibel smooth and transversal to C' and E'. Then $C = C + nH - nH \sim C' - E'$ and $D = D + nH - nH \sim D' - F'$, i.e. we can write both C and D up to linear equivalence as a difference of two irreducible smooth curves with transversal intersections. By properties (1)-(4) then

$$\begin{aligned} C.D &= (C' - E').(D' - F') = C'.D' - E'.D' - C'.F' + E'.F' \\ &= \#(C' \cap D') - \#(E' \cap D') - \#(C' \cap F') + \#(E' \cap F') \end{aligned}$$

which shows that if such a pairing exists (i.e. is well-defined), it must be given by the above formula.

Assume now C, D are very ample divisors on X. By Lemma 4.4 there are irreducible smooth $C' \in |C|$, $D' \in |D|$ which intersect transversally. We claim that

$$C.D = C'.D' = \#(C' \cap D')$$

is well defined and satisfies (1)-(4). Assume $D'' \in |D|$ is again irreducible smooth and intersects C' transverally. From Lemma 4.5, since $D' \sim D''$, we have

$$\#(C' \cap D') = \deg(\mathcal{L}(D') \otimes \mathcal{O}_{C'}) = \deg(\mathcal{L}(D'') \otimes \mathcal{O}_{C'}) = \#(C' \cap D'');$$

and likewise replacing C' by a C'' we see that

$$\#(C' \cap D') = \#(C'' \cap D''),$$

i.e. $C.D$ is well defined. Using elementary properties of line bundles on a curve it is easy to see that (1)-(4) hold in this setting.

To define the pairing for any two divisors C, D set (with the above notations and choices of C', D', E' and F')

$$C.D = \#(C' \cap D') - \#(E' \cap D') - \#(C' \cap F') + \#(E' \cap F').$$

By the pervious paragraph each of the terms in this formula is well defined and we need to show that the entire formula is well defined. By construction, $C \sim C' - E'$ and $D \sim D' - F'$. Assume we have similarly $C \sim C'' - E''$ and $D \sim D'' - F''$. Then $C' + E'' \sim C'' + E'$ and since these diviors are very ample, we have from the above

$$C'.D' - E'.D' = C''.D' - E''.D'.$$

An analogous argument, intersecting $D' + F'' \sim D'' + F'$ with C', shows

$$C'.D' - C'.F' = C'.D'' - C'.F'',$$

i.e. the formula for $C.D$ is well defined and has the claimed properties. □

If C, D are curves without common components, the intersection product can be computed without 'moving' the curves.

Proposition 4.6. *Let C, D be curves on a surface X without common irreducible components. Then $C.D = \sum_{P \in C \cap D} (C.D)_P$.*

Proof. As in the proof of Lemma 4.5 we have the exact sequence

$$0 \to \mathcal{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0,$$

where $\mathcal{O}_{C\cap D}$ is a skyscraper sheaf which is supported on the finite set $C \cap D$; for each $P \in C \cap D$ we have $(\mathcal{O}_{C\cap D})_P = \mathcal{O}_{X,P}/(f, g)$. Hence

$$\chi(\mathcal{O}_C) = \dim_k \mathrm{H}^0(X, \mathcal{O}_{C\cap D}) = \sum_{P \in C \cap D} (C.D)_P$$

On the other hand, the addivity of the Euler characteristic implies that

$$\chi(\mathcal{O}_{C\cap D}) = \chi(\mathcal{O}_C) - \chi(\mathcal{L}(-D) \otimes \mathcal{O}_C),$$

where the right hand side only depends on the linear equivalence class of the divisor D; by symmetry this also holds for the divisor C. As in the proof of Theorem 4.1 we may replace C and D with the differences of two irreducible two smooth curves with transversal intersection. Since the intersection pairing is additive, we may assume C and D are irreducible smooth curves with transversal intersection. Then $\chi(\mathcal{O}_C) = 1 - g(C)$ and by Riemann-Roch 3.3 for curves we have $\chi(\mathcal{L}(-D)) = \deg(\mathcal{L}(-D) \otimes \mathcal{O}_C) + 1 - g(C)$. Therefore

$$\chi(\mathcal{O}_{C\cap D}) = -\deg(\mathcal{L}(-D) \otimes \mathcal{O}_C) = \deg(\mathcal{L}(D) \otimes \mathcal{O}_C) = \#(C \cap D),$$

where the last identity results from Lemma 4.5. □

Examples 4.7. (a) If D is any divisor on a surface X, we have its self-intersection number $D^2 = D.D$. This integer cannot be calculated using the method of Proposition 4.6, even when D is smooth. For an irreducible smooth curve C on X the proof of Lemma 4.5 shows that $C^2 = C.C = \deg(\mathcal{L}(C) \otimes \mathcal{O}_C)$. If $\mathcal{L}(-C)$ is the line bundle associated to C, there is an isomorphism $\mathcal{L}(-C) \otimes \mathcal{O}_C \cong \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of the closed subscheme C on X. Thus $\mathcal{L}(C) \otimes \mathcal{O}_C = (\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{N}_{C/X}$ is the normal sheaf of C on X, and $C^2 = \deg(\mathcal{N}_{C/X})$.

(b) Let $X = \mathbb{P}^2_k$. Then $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by a line H. Since any two lines are linearly equivalent and any two distinct lines meet in one point, we have $H^2 = 1$. If $C, D \subseteq \mathbb{P}^2_k$ are curves of degrees m and n, then $C \sim mH$ and $D \sim nH$. Hence $C.D = mH.nH = mnH^2 = mn$; this is one proof of Bezout's Theorem.

(c) Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k \cong Q$, where $Q \subseteq \mathbb{P}^3_k$ is the smooth quadric given by $xy = uv$. Then $\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$ generated by a line L of type $(1,0)$ and a line M of type $(0,1)$. Then $L^2 = M^2 = 0$ and $L.M = 1$ since two lines in the same family are skew and two lines in different families meet in one point. If C and D are divisor on X of type (a,b) and (c,d), then $C.D = (aL + bM).(cL + dM) = ad + bc$.

(d) For a surface X we have the canonical sheaf $\omega_X = \wedge^2\Omega^1_X = \Omega^2_X$; a canonical divisor $K = K_X$ is a divisor on X such that $\mathcal{L}(K_X) = \omega_X$. The self-intersection K^2 is a number which only depends on X. For example, if $X = \mathbb{P}^2_k$, then with the notations from (b) above, $K = -3H$, i.e. $K^2 = 9$.

(e) Let $\pi : \widetilde{X} \to X$ be the blow up of a (smooth) surface X at a point $P \in X$, and let $E = \pi^{-1}(P)$ be the exceptional divisor. Then $E \cong \mathbb{P}^1_k$ and

$$E^2 = \deg(\mathcal{N}_{E/X}) = \deg(\mathcal{L}(E) \otimes \mathcal{O}_E) = \deg(\mathcal{O}_{\mathbb{P}^1_k}(-1)) = -1.$$

This negative self intersection number means that we cannot find a divisor D on X, which is linearly equivalent to E but different from E, i.e. we cannot 'move' E in its linear equivalence class.

Proposition 4.8. *(Adjunction Formula) Let C be a smooth curve of genus g on a surface X. if K_X is a canonical divisor, then*

$$C.(C + K_X) = 2g - 2$$

Proof. By [2, II, Proposition 8.20] $\omega_C \cong \omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$. By Corollary 3.4 $\deg(\omega_C) = 2g(C) - 2$ and by Lemma 4.5 $\deg(\omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C) = C.(C + K_X)$. □

Examples 4.9. (a) Let $\pi : \widetilde{X} \to X$ be the blow up of a point on a smooth surface from Example 4.7(e). Then $E \cong \mathbb{P}^1_k$ and $E^2 = -1$. The Adjunction Formula 4.8 implies that $E.(E + K_{\widetilde{X}}) = E^2 + E.K_{\widetilde{X}} = 2g(E) - 2 = -2$, therefore $E.K_{\widetilde{X}} = -1$.

(b) Let $X = \mathbb{P}^2_k$ and let $C \subseteq X$ be a smooth projective curve of degree d. If $H \subseteq \mathbb{P}^2_k$ is a line which generates $\mathrm{Pic}(X)$, then $C \sim dH$ and $K_X = -3H$, see Examples 4.7(b) and (d). The Adjunction Formula 4.8 implies so

$$g(C) = 1 + \frac{C^2 + C.K_{\mathbb{P}^2}}{2} = 1 + \frac{d^2 - 3d}{2} = \frac{1}{2}(d-1)(d-2).$$

If D is a divisor on a surface X, the superabundance of D is defined as $s(D) = \dim_k \mathrm{H}^1(X, \mathcal{L}(D))$; by definition $s(D)$ is a non-negative integer.

Theorem 4.10. *(Riemann-Roch for surfaces) Let D be any divisor on a (smooth projective) surface X. Then*

$$l(D) - s(D) + l(K_X - D) = \frac{1}{2}D.(D - K_X) + \chi(\mathcal{O}_X).$$

Proof. As before $\mathcal{L}(K_X - D) \cong \mathcal{L}(D)^\vee \otimes \omega_X$ and by Serre Duality we have

$$l(K_X - D) = \dim_k \mathrm{H}^0(X, \mathcal{L}(D)^\vee \otimes \omega_X) = \dim_k \mathrm{H}^2(X, \mathcal{L}(D),$$

i.e. the left hand side of the above equation is $\chi(\mathcal{L}(D))$.

Since both sides of this equation only depend on the linear equivalence class of D we map replace D as in the proof of Theorem 4.1 with the difference $C - E$ of two irreducible smooth curves. Consider the ideal sheaves $\mathcal{L}(-C)$ and $\mathcal{L}(-E)$ of C and E. From the exact sequence of sheaves on X

$$0 \to \mathcal{L}(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

we obtain by tensoring with $\mathcal{L}(C)$ the exact sequence

$$0 \to \mathcal{L}(C - E) \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_E \to 0.$$

Similarly, tensoring the exact sequence

$$0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

with $\mathcal{L}(C)$ we have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_C \to 0.$$

Since the Euler characteristic χ is additive we obtain from these sequences

$$\chi(\mathcal{L}(C - E)) = \chi(\mathcal{L}(C) \otimes \mathcal{O}_C) - \chi(\mathcal{L}(C) \otimes \mathcal{O}_E) + \chi(\mathcal{O}_X).$$

By Riemann-Roch 3.3 for curves, together with Lemma 4.5, then

$$\begin{aligned} \chi(\mathcal{L}(C) \otimes \mathcal{O}_C) &= C^2 + 1 - g(C), \\ \chi(\mathcal{L}(C) \otimes \mathcal{O}_E) &= C.E + 1 - g(E), \end{aligned}$$

where we can compute the genera from the Adjunction Formula 4.8

$$\begin{aligned} g(C) &= \tfrac{1}{2}C.(C + K_X) + 1 \\ g(E) &= \tfrac{1}{2}E.(E + K_X) + 1. \end{aligned}$$

In summary, we have shown that

$$\begin{aligned} \chi(\mathcal{L}(D) &= \chi(\mathcal{L}(C - E)) = \tfrac{1}{2}(C - E)(C - K_X) + \chi(\mathcal{O}_X) \\ &= \tfrac{1}{2}D.(D - K_X) + \chi(\mathcal{O}_X). \end{aligned}$$

□

If X is a smooth projective curve and D is a divisor with the property that $\deg(D) > 2g(X) - 2$, then $\dim_k \mathrm{H}^1(X, \mathcal{L}(D)) = 0$: By Corollary 3.4 $\deg(K_X) = 2g(X) - 2$; hence $\deg(K_X - D) < 0$ and by Lemma 3.2 $l(K_X - D) = 0$. Now Serre Duality implies $\dim_k \mathrm{H}^1(X, \mathcal{L}(D)) = 0$.

We want to prove an analogous result for surfaces. For this we need the following observation (cp. [2, V, Exercise 1.2]): If H is a very ample divisor on a surface X and C is any curve (i.e. effective divisor) on X, the intersection number $C.H$ is the degree of C in the projective embedding given by H. In particular, it is positive.

Lemma 4.11. *Let H be an ample divisor on a surface X. Then there is an integer n_0 such that if D is a divisor on X with $D.H > n_0$, then* $\mathrm{H}^2(X, \mathcal{L}(D)) = 0$.

Proof. Let $m > 0$ be an integer such that mH is very ample (cp. [2, II, Theorem 7.6]). If $\mathrm{H}^2(X, \mathcal{L}(D)) \neq 0$, by Serre Duality $l(K_X - D) > 0$, and $K_X - D$ is effective. By the remark above $(K_X - D).mH = m[(K_X.H) - (D.H)] > 0$, hence $D.H < K_X.H$. If $n_0 = K_X.H$, then $D.H > n_0$ and therefore $l(K_X - D) = 0$. □

Lemma 4.12. *Let H be an ample divisor on a surface X and let D be a divisor on X such that $D.H > 0$ and $D^2 > 0$. Then for $n >> 0$ the divisor nD is linearly equivalent to an effective divisor.*

Proof. Choose n_0 as in Lemma 4.11 such that for any divisor D' with $D'.H > n_0$ we have $l(K_X - D') = 0$. Since $D.H > 0$ by assumption, for $n >> 0$ then $nD.H > n_0$ and therefore $l(K_X - nD) = \dim_k \mathrm{H}^2(X, \mathcal{L}(nD)) = 0$. Since $s(nD) = \dim_K \mathrm{H}^1(X, \mathcal{L}(nD)) \geq 0$, it follows from Riemann-Roch 4.10 that

$$l(\mathcal{L}(nD)) \geq \frac{1}{2}n^2D^2 - \frac{1}{2}nD.K_X + \chi(\mathcal{O}_X).$$

Since $D^2 > 0$ the right hand side becomes positive for $n >> 0$, hence for large enough n the divisor nD is linearly equivalent to an effective divisor (in fact $l(\mathcal{L}(nD)) \to \infty$ as $n \to \infty$). □

Definition 4.13. A divisor D on a surface X is numerically equivalent to zero, $D \equiv 0$, if $D.E = 0$ for all divisors E on X. Two divisors D, E are numerically equivalent, $D \equiv E$, if $D - E \equiv 0$. Let $\mathrm{Pic}^0(X) \subseteq \mathrm{Pic}(X)$ be the subgroup of all line bundles $\mathcal{L}(D)$ such that $D \equiv 0$. The quotient $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the Neron-Severi group of X.

Theorem 4.14. *(Hodge Index Theorem) Let H be an ample divisor on a surface X. If D is a divisor on X such that $D \not\equiv 0$ and $D.H = 0$, then $D^2 < 0$.*

NB. If H is an ample divisor on a surface X, then $H^2 > 0$ (the pullback of an ample line bundle on a projective variety to a curve on X has positive degree). There is a decomposition $\mathrm{NS}(X) \otimes \mathbb{Q} \cong \mathbb{Q}\{H\} \oplus H^\perp$ and the Hodge Index Theorem says the intersection pairing is negative definite on $H^\perp$, i.e. for $D \in H^\perp$ we have $D^2 < 0$. Equivalently: if $\rho(X) = \dim_{\mathbb{Q}} \mathrm{NS}(X) \otimes \mathbb{Q}$ (the Neron-Severi group $\mathrm{NS}(X)$ is a finitely generated abelian group), then the signature (or index) of the intersection pairing on $\mathrm{NS}(X) \otimes \mathbb{Q}$ is $(1, \rho(X) - 1)$.

Proof. Assume first $D^2 > 0$. Set $H' = D + nH$. Then for $n >> 0$ H' is ample (cp. proof of Theorem 4.1) and $D.H' = D^2 > 0$. By Lemma 4.12 mD is linearly equivalent to an effective divisor for all $m >> 0$. Let $k > 0$ be such that kH is very ample. Then $mD.kH = mk(D.H) > 0$, thus $D.H > 0$, which contradicts our assumptions. If $D^2 = 0$, then $D \not\equiv 0$ implies that there is a divisor E on X such that $D.E > 0$. For $E' = (H^2)E - (E.H)H$ we have $E'.D = (H^2)E.D > 0$ and $E'.H = 0$, i.e. replacing E by E' we may assume $E.H = 0$. Set $D' = nD + E$. Then $D'.H = 0$ and $(D')^2 = 2nD.E + E^2$. Because $D.E > 0$ there is some $n >> 0$ such that $(D')^2 > 0$ and applying the previous argument to D' gives again a contradiction. $\square$

Remark 4.15. Assume X is a smooth projective surface over $\mathbb{C}$. Consider X as a complex projective manifold of complex dimension 2 and let $\mathcal{O}_{an}$ be the sheaf of holomorphic functions on X. The map $e : \mathcal{O}_{an} \to \mathcal{O}_{an}^{\times}$, $f \mapsto exp(2\pi i f)$ is locally surjective and we have the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_{an} \to \mathcal{O}_{an}^{\times} \to 0.$$

In the resulting long exact sequence of (singular) cohomology groups we have $\mathrm{H}^0(X, \mathcal{O}_{an}) = \mathbb{C}$ and $\mathrm{H}^0(X, \mathcal{O}_{an}^{\times}) = \mathbb{C}/\mathbb{Z}$ and the map between these groups is the obvious surjection. Consider the exact sequence

$$0 \to \mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_{an}) \to \mathrm{H}^1(X, \mathcal{O}_{an}^{\times}) \xrightarrow{c_1} \mathrm{H}^2(X, \mathbb{Z})$$

By a Theorem of Serre $\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathcal{O}_{an}^{\times})$ and one may identify the image of c_1 with the Neron-Severi group $\mathrm{NS}(X)$. In particular, we see that $\mathrm{Pic}(X)$ sits in the exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{O}_{an}) / \mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where the left hand side is a complex torus and the right hand side is a finitely generated group. In this setting we may identify $C.D = c_1(C) \cup c_1(D)$, i.e. the intersection product on divisors corresponds to the cup product in singular cohomology. In this context the Hodge Index Theorem is a classical result from Kähler geometry: Let X be a complex projective surface and ω the class of an ample line bundle. Then the cup product is negative definite on $\omega^{\perp} \subseteq \mathrm{H}^{1,1}(X, \mathbb{C}) \cap \mathrm{H}^2(X, \mathbb{R})$.

5. Monoidal Transformations

We use the convetions from the pervious section: A surface X is a smooth projective surface, a curve on a surface is an effective divisor, and a point on a surface is a closed point.

A monoidal transformation on a surface X is the birational morphism which results from blowing up a single point P on X. We will eventually show that any birational transformation (i.e. birational morphism) between two projective surfaces can be factored in a finite sequence of monoidal operations and their inverses; thus monoidal transformation are central to the study of surfaces.

Let X be a surface, $P \in X$ a point and let $\pi : \widetilde{X} \to X$ the monoidal transformation with center P, i.e. the blow up of X at P. Thus π induces an isomorphism from $\widetilde{X} \setminus \pi^{-1}(P)$ onto $X \setminus P$. The inverse image $E = \pi^{-1}(P)$ is the exceptional curve on $\widetilde{X}$. We know that $E \cong \mathbb{P}^1_k$, and that the self-intersection number of E on $\widetilde{X}$ is $E^2 = -1$, see Example 4.7(e).

Proposition 5.1. *Let X be a surface and $\pi : \widetilde{X} \to X$ the monoidal transformation with center P. Then the two maps $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X})$, $1 \mapsto 1 \cdot |\mathcal{L}(E)|$ (or $1 \mapsto 1 \cdot |E|$) induces an isomorphism*

$$(\#) \quad \mathrm{Pic}(X) \times \mathbb{Z} \stackrel{\cong}{\to} \mathrm{Pic}(\widetilde{X}),\ (D, n) \mapsto \pi^* D + nE.$$

Let $\pi_* : \mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z} \to \mathrm{Pic}(X)$, $\pi^* D + n\mathbb{Z} \ \mapsto D$ be the projection onto the first factor under the isomorphism $(\#)$.

The intersection pairing on $\widetilde{X}$ is determined by

(a) $C, D \in \mathrm{Pic}(X) \Rightarrow \pi^* C.\pi^* D = C.D$,
(b) $C \in \mathrm{Pic}(X) \Rightarrow \pi^* C.E = 0$,
(c) $E^2 = -1$.
(d) $C \in \mathrm{Pic}(X), D \in \mathrm{Pic}(\widetilde{X}) \Rightarrow \pi^* C.D = C.(\pi_* D)$.

Proof. For any proper closed subset $Z \subseteq X$ and $U = X \setminus Z$ there is a surjective restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$, $\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$; this map is an isomorphism if $\mathrm{codim}_X(Z) \geq 2$ and for Z irreducible of codimension 1 yields the exact sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$, where the first map is $1 \mapsto 1 \cdot |Z|$; for details see [2, II, Proposition 6.5]. Since X and U are smooth we have isomorphism $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ and $\mathrm{Cl}(U) \cong \mathrm{Pic}(U)$ and the fact that P has codimension 2 implies that $\mathrm{Pic}(X) \cong \mathrm{Pic}(X \setminus P)$. On the other hand, $\widetilde{X} \setminus E \cong X \setminus P$, hence there is an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X) \to 0,$$

where the first map is $1 \mapsto 1 \cdot |\mathcal{L}(E)|$. Assume $|\mathcal{L}(E)| = 0$. Then for $n > 0$

$$0 = n|\mathcal{L}(E)|.n|\mathcal{L}(E)| = n^2 E^2 = -n^2,$$

Widerspruch. Hence $\mathbb{Z} \to \mathrm{Pic}(\widetilde{X})$ is injective and since π^* splits this sequence we have the isomorphism $(\#)$.

For (a) and (b) we may write C and D as in the proof of Theorem 4.1 as the difference of two irreducible smooth curves $C \sim C' - E'$ and $D \sim D' - F'$ with all intersections transversal. This uses the Moving Lemma 4.4 which allows us to also assume that these smooth curves do not meet P. Hence π^* does not affect the intersection (for an irreducible curve $|\pi^* C| = |\pi^{-1}(C)|$) which gives (a). Similary, $\pi^* C$ does not meet E, thus $\pi^* C.E = 0$, i.e. we have (b). The same argument proves (d) (write C as a product of irreducible smooth curves not containing P); (c) we have shown in Example 4.7(e). $\square$

Lemma 5.2. *Let $\pi : \widetilde{X} \to X$ be as above. Then we have in Pic($\widetilde{X}$)*

$$K_{\widetilde{X}} = \pi^* K_X + E;$$

furthermore $K_{\widetilde{X}}^2 = K_X^2 - 1$.

NB. The second formula says that the self-intersection number of a surface is not a birational invariant. For example, if $\widetilde{X}$ is the blow up of a point on $X = \mathbb{P}_k^2$, then $K_X^2 = 9$ by Example 4.7(d) and $K_{\widetilde{X}}^2 = 8$ by the above formula.

Proof. From Proposition 5.1 we have an isomorphism

$$\operatorname{Pic}(X) \oplus \mathbb{Z} \to \operatorname{Pic}(\widetilde{X}),\ (D, n) \mapsto \pi^* D + nE.$$

If $\widetilde{X}$ is the monoidal transformation with center P and exceptional curve E, then $\widetilde{X} \setminus E \cong X \setminus P$ and from this it is easy to see (cp. proof of Proposition 5.1) that $K_{\widetilde{X}} = \pi^* K_X + nE$ for some $n \in \mathbb{Z}$. We show $n = 1$. From the Adjunction Formula 4.8, applied to $E \subseteq \widetilde{X}$ and using Proposition 5.1 (b),(c)

$$-2 = 2g(E) - 2 = E.(E + K_{\widetilde{X}}) = E.(E + \pi^* K_X + nE) = -1 - n;$$

hence $n = 1$. For the second formula note that by Proposition 5.1(a),(b),(c)

$$K_{\widetilde{X}}^2 = (\pi^* K_X + E).(\pi^* K_X + E) = K_X^2 + E^2 = K_X^2 - 1.$$

□

Proposition 5.3. *Let $\pi : \widetilde{X} \to X$ be a above. Then $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $\mathrm{H}^i(X, \mathcal{O}_X) \cong \mathrm{H}^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ for all $i \geq 0$.*

Proof. The natural proof of this result uses spectral sequences and will be omitted here. For an alternate argument, see [2, V, Proposition 3.4]. □

Corollary 5.4. *For a monoidal transformation $\pi : \widetilde{X} \to X$ we have $p_a(\widetilde{X}) = p_a(X)$, i.e. the arithmetic genus p_a is invariant under monoidal transformations.*

NB. It also follows from Corollary 5.3 that $\widetilde{X}$ and X have the same geometric genus $p_g(X) = \dim_k \mathrm{H}^2(X, \mathcal{O}_X)$ and the same irregularity $q(X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X)$.

Proof. There are various ways to define the arithmetic genus, for example, if X is irreducibel of dimension r and $\mathrm{H}^0(X, \mathcal{O}_X) = k$ (X projective) as

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_K H^{r-i}(X, \mathcal{O}_X);$$

see [2, III, Exercise 5.3]. The claim follows using this definition from Proposition 5.3 □

Let X be a surface and $\pi : \widetilde{X} \to X$ a monoidal transformation with center P. If C is a curve (=effective divisor) on X such that $P \notin C$, then $\pi^{-1}(C) \cong C$. If $p \in C$, then $\pi^{-1}(C)$ contains E, and $\pi^{-1}(C \setminus P) = \pi^{-1}(C \cap (X \setminus P))$ is a curve minus a point on $\widetilde{X}$. The strict transform $\widetilde{C}$ of C is the closure of $\pi^{-1}(C \setminus P)$ in $\widetilde{X}$, i.e. $\widetilde{C} = \overline{\pi^{-1}(C \setminus P))} \subseteq \widetilde{X}$.

Proposition 5.5. *Let X be a surface and $\pi : \widetilde{X} \to X$ a monoidal transformation with center P. Assume C is a curve on X which meets P with multiplicity $r \geq 1$ (thus $P \in C$ and C is smooth at $P \Leftrightarrow r = 1$). Then*

$$\widetilde{C} = \pi^* C - r\mathbb{Z}.$$

Proof. See [2, V, Proposition 3.6]. □

Corollary 5.6. *With the same hypothesis we have* $\widetilde{C}.E = r$ *and*

$$p_a(\widetilde{C}) = p_a(C) - \frac{1}{2}r(r-1).$$

Proof. By Proposition 5.5 $\widetilde{C} = \pi^*C - rE$ and by Proposition 5.1

$$\widetilde{C}.E = \pi^*C.E - r(E.E) = 0 - r(-1) = r.$$

If D is a projective scheme of dimension 1, the arithmetic genus is defined as $p_a(D) = 1 - \chi(\mathcal{O}_D)$; if D is an effective divisor on a surface X, then $2p_a(D) - 2 = D.(D + K_X)$; see [2, V, Exercise 1.3]. Therefore, using Proposition 5.1 and Lemma 5.2, we have

$$\begin{aligned} 2p_a(\widetilde{C}) - 2 &= \widetilde{C}.(\widetilde{C} + K_{\widetilde{X}}) = (\pi^*C - rE).(\pi^*C - rE + (\pi^*K_X + E)) \\ &= 2p_a(C) - 2 - r(r-1), \end{aligned}$$

which gives the formula $p_a(\widetilde{C}) = p_a(C) - \frac{1}{2}r(r-1)$. □

Proposition 5.7. *Let C be an irreducible curve on a surface X. Then there exists a finite sequence of monoidal transformations with suitable centers*

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

such that the strict transform C_n of C in X_n is smooth.

Proof. If C is already smooth, set $n = 0$. Otherwise there is a singular point P on C with multiplicity $r \geq 2$. Let $X_1 \to X$ be the monoidal transformation with center P and let C_1 be the strict transform of C in X_1. By Corollary 5.6 we have $p_a(C_1) < p_a(C)$. If C_1 is smooth, we are done. Otherwise choose a singular point P_1 on C_1 and iterate. This gives a sequence of monoidal transformations $\cdots \to X_n \to \cdots \to X_0 = X$ with $p_a(C_i) < p_a(C_{i-1})$ for all i. Since the arithmetic genus of any irreducible curve is non-negative ($p_a(C) = \dim_k \mathrm{H}^1(C, \mathcal{O}_C)$; see [2, III, Exercise 5.3]) this process terminates and for some n the strict transform C_n is smooth. □

6. Birational Transformations

We show the following 'Factorization Theorem' for (smooth projective surfaces): If $T : X' \dashrightarrow X$ is a birational transformation of (smooth projective) surfaces, then T factors as a finite sequence of monoidal transformations and their inverses. This implies, using Corollary 5.4, that the arithmetic genus p_a is a birational invariant for (smooth projective) surfaces.

Example 6.1. For any smooth projective surface X we have $p_a(X) = 1 - \dim_k \mathrm{H}^1(X, \mathcal{O}_X)$, see proof of Corollary 5.4. If $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, then $X \sim \mathbb{P}^2_k$ (i.e. X is rational) and therefore $p_a(X) = p_a(\mathbb{P}^2_k) = 1$. Let E be an elliptic curve and $Y = E \times \mathbb{P}^1_k$. Assume $Y \sim \mathbb{P}^2_k$. Then $p_a(Y) = 1$. The projection $p : Y \to E$ induces an injective k-linear map $p^* : \mathrm{H}^1(E, \mathcal{O}_E) \to \mathrm{H}^1(Y, \mathcal{O}_Y)$. Since $\dim_k \mathrm{H}^1(E, \mathcal{O}_E) = p_a(E) = g(E) = 1$, we have $p_a(Y) < 1$ and therefore $Y \not\sim \mathbb{P}^2_k$.

• Let $f : X \dashrightarrow Y$ be a birational transformation of projective varieties (of any dimension) and let $\emptyset \neq U \subseteq X$ be an open subset such that $f|_U = \phi : U \to Y$ is a morphism. If $\emptyset \neq V \subseteq X$ is another such open set and $f|_V = \psi : V \to Y$ is the corresponding morphism, then ϕ and ψ agree where both are defined and glue to a morphism $U \cup V \to Y$ [2, I, Exercise 4.2]. Hence there is a largest open set $\emptyset \neq U \subseteq X$ on which f is represented by a morphism (f is 'defined' on U). The fundamental points of f are the elements of $X \setminus U$. For example, if $\pi : \widetilde{X} \to X$ is the blow up of a point P on a (smooth projective) surface X, then π has no fundamental points and the only fundamental point of π^{-1} is P.

• Let $f : X \dashrightarrow Y$ be a birational transformation which is defined on $U \subseteq X$ and let $f|_U = \phi : U \to Y$ be the corresponding morphism. Write $\Gamma_\phi = \{(u, \phi(u) \mid u \in U\} \subseteq U \times Y$ for the graph of ϕ. The closure Γ of Γ_ϕ in $X \times Y$ is the graph of f. Let $p_1 : \Gamma \to X$ and $p_2 : \Gamma \to Y$ be the projections. If $Z \subseteq X$ is a subset, the total transform of Z (under f) is $f(Z) = p_2(p_1^{-1}(Z)) \subseteq Y$. If f is defined at $P \in X$, then $f(P) = p_2((P, \phi(P)) = \phi(P)$; if P is a fundamental point of f, then $f(P)$ consists of more than one element.

We will need the following two results:

Lemma 6.2. *Let $f : X \dashrightarrow Y$ be a birational transformation of projective varieties with X normal. Then the fundamental points of X form a closed subset of codimension ≥ 2.*

NB. If X is a smooth projective surface and $f : X \dashrightarrow Y$ is a birational transformation with Y projective the Lemma states, that the fundamental points of f are a finite set of closed points.

Proof. [2, V, Lemma 5.1]. □

Theorem 6.3. *(Zariski's Main Theorem) Let $f : X \dashrightarrow Y$ be a birational transformation of projective varieties with X normal. If P is a fundamental point of f, then the total transfom $f(P)$ is connected of dimension ≥ 1.*

NB. If X and Y are smooth projective surfaces, $f : X \dashrightarrow Y$ is a birational transformation and P is a fundamental point of f, then Theorem 6.3 implies that $f(P)$ contains an irreducible curve $D \subseteq Y$: If $f(P)$ has dimension 2, then $f(P) = Y$ and since the degree map $\mathrm{Pic}(Y) \to \mathbb{Z}$ is surjective, there is an irreducible curve $D \subseteq Y$ of degree 1. If $f(P)$ has dimension 1, let D be an irreducible component of $f(P) \subseteq Y$.

Proof. [2, V, Theorem 5.2]. □

Theorem 6.4. *(Factorization Theorem for Surfaces) Let $f : X' \dashrightarrow X$ be a birational transformation of (smooth projective) surfaces. Then f can be factored as a finite sequence of monoidal transformations and their inverses.*

Corollary 6.5. *For a smooth projective surface Y the following are birational invariants: geometric genus $p_g(Y) = \dim_k \mathrm{H}^2(Y, \mathcal{O}_Y)$, arithmetic genus $p_a(Y) = \sum_{i=0}^{1}(-1)^i \dim \mathrm{H}^i(Y, \mathcal{O}_Y)$, irregularity $q(Y) = \dim_k \mathrm{H}^1(Y, \mathcal{O}_Y)$.*

Proof. (of Corollary 6.5) Follows from Theorem 6.4, combined with Proposition 5.3 and Corollary 5.4. $\square$

Lemma 6.6. *Let $T : X \dashrightarrow X'$ be a birational transformation of (smooth projective) surfaces. Then there exists a (smooth projective) surface X'', together with biartional morphisms $f : X'' \to X$ and $g : X'' \to X$ such that $T = g \circ f^{-1}$.*

Proof. Let H' be a very ample divisor on X' and let $C' \in |2H'|$ be an irreducible smooth curve which does not pass through the (finitely many) fundamental points of T^{-1}. Let $\emptyset \neq U' \subseteq X'$ be the open set on which T^{-1} is defined and represented by a morphism $T^{-1}|_{U'} = \phi : U' \to X$. Then $C' \subseteq U'$; let $C = \phi(C') \subseteq X$ be the image of C' in X. Consider the integer

$$(\#) \quad m(T) = p_a(C) - p_a(C').$$

By [2, IV, Exercise 1.8] we have

$$m(T) \geq 0 \text{ and } m(T) = 0 \Leftrightarrow C \cong C'.$$

Choose another such curve $C_1' \in |2H'|$ (i.e. C_1' an irreducible smooth curve which does not meet the fundamental points of T^{-1}) and let $C_1 = \phi(C_1')$. Because $C' \sim C_1'$ there is a rational function $f \in k(X')^\times$ such that $\operatorname{div}(f) = C' - C_1'$. Let U be an open set such that $\phi : U' \to U$ is an isomorphism and view $f \in k(X')^\times \cong k(U')^\times$ as an element of $k(X)^\times \cong k(U)^\times$. Then $\operatorname{div}(f) = C - C_1$. Since the arithmetic genus of a curve only depends on the linear equivalence class [2, V, Exercise 3.2] the integer $m(T)$ in (#) depends only on T and H' but not on the choice of $C' \in |2H'|$.

Let $C' \in |2H'|$ be such an irreducible smooth curve which does not meet the fundamental points of T^{-1}. If $m(T) > 0$, then C must be singular (if C is smooth, then $C' \cong C$ and $m(T) = 0$). Let $P \in C$ be a singular point, $\pi : \widetilde{X} \to X$ the monoidal transformation with center P, and $\widetilde{C}$ the strict transform of C under π. Because P is singular is a singular point on C it has multiplicity $r \geq 2$ and by Corollary 6.5 $p_a(\widetilde{C}) < p_a(C)$. Hence if $\widetilde{T} = T \circ \pi$, we have $m(\widetilde{T}) < m(T)$. Iteration yields a (smooth projecvtive) surface X'' and a birational morphism $f : X'' \to X$ which is a finite sequence of monoidal transformations such that if $T' = T \circ f$, then $m(T') = 0$.

We show T' is a morphism. If this holds, then $g = T' : X'' \to X'$ will have the claimed properties. Assume T' is not a morphism. Then it has a fundamental point $P'' \in X''$. By Zariski's Main Theorem 6.3 the total transform $T'(P'') \subseteq X'$ contains an irreducible curve $E' \subseteq X'$. Because H' is very ample, $H'.E' > 0$ (see remark before Lemma 4.11). Choose an irreducible smooth curve $C' \in |2H'|$ such that C' and E' meet transversally and C' does not meet any fundamental points of T^{-1}. Then $C'.E' = 2H'.E' = 2(H'.E') \geq 2$, i.e. $C' \cap E'$ contains at least two distinct points. Let C be the image of C' in X''. Then P'' is at least a double point on C, thus C is not smooth and therfore $m(T') > 0$; this contradiction proves the Lemma. $\square$

Lemma 6.7. *Let $f : X' \dashrightarrow X$ be a birational morphism of (smooth projective) surfaces and let $P \in X$ be a fundamental point of T^{-1}. Then f factors through the monoidal transformation $\pi : \widetilde{X} \to X$ with center P, i.e. the birational transformation $\pi^{-1} \circ f$ is a morphism.*

Proof. (sketch) We show $T = f \circ \pi^{-1} : X \dashrightarrow \widetilde{X}$ is a morphism. Assume not. Then T has a fundamental point$P' \in X'$. Obviously $f(P') = P$ (else $T(P')$ is defined). By Zariski's Main Theorem 6.3 $T(P')$ is connected of dimension ≥ 1, i.e. $T(P') = E$, the exceptional curve on $\widetilde{X}$. By Lemma 6.2 T^{-1} is defined on all but finitely many points, hence there is a point $Q \in E$ such that $P' = T^{-1}(Q)$. A local computation, involving the equations defining $\widetilde{X}$, gives a contradiction. Hence T is a morphism. $\square$

Let $f : X' \to X$ be a birational morphism of (smooth projective) surfaces and let $n(f)$ be the number of irreducible curves $C' \subseteq X'$ such that $f(C')$ is a point. Then $n(f)$ is finite: If $f(C') = P$ is a point, then P is a fundamental point of f^{-1} and by Lemma 6.2 there is only a finite number of fundamental points. For each fundamenal point P the inverse image $f^{-1}(P) \subseteq X'$ is a closed subset which has only a finite number of irreducible components. Thus $n(f)$ is finite.

Lemma 6.8. *Let $f : X' \to X$ be a birational morphism of (smooth projective) surfaces. Then f can be factored as a composition of $n(f)$ monoidal transformations.*

Proof. Assume $P \in X$ is a fundamental point of f^{-1}. If $\pi : \widetilde{X} \to X$ is the monoidal transformation with center P, $f_1 = \pi^{-1} \circ f : X \to \widetilde{X}$ is a birational morphism by Lemma 6.7. We claim $n(f_1) = n(f) - 1$: If $f_1(C')$ is a point, then $f(C')$ is a point. Conversely, if $f(C')$ is a point, then either $f_1(C')$ is a point or $f_1(C') = E$, the exceptional curve on $\widetilde{X}$. Since f_1^{-1} has only a finite number of fundamental points, there is a unique irreducible curve $E' \subseteq X'$ such that $f_1(E') = E$ and $n(f_1) = n(f) - 1$.

It follows form the above that after $m = n(f)$ sucessive blow ups one obtains a birational morphism $f_m : X' \to X_m$ with $n(f_m) = 0$. By Zariski's Main Theorem 6.3 f_m^{-1} has no fundamental points and is therefore an isomorphism. $\square$

Proof. (of Theorem 6.4) Let $f : X' \dashrightarrow X$ be a birational transformation. By Lemma 6.6 there is a smooth projective surface X'', together with birational morphisms $g : X'' \to X'$ and $h : X'' \to X$ such that $h = f \circ g$. By Lemma 6.8 g (resp. h) can be factor as a composition of $n(g)$ (resp. $n(h)$) monoidal transformation. This proves the Theorem. $\square$

Let $\pi : \widetilde{X} \to X$ be the blow up of a point P on a (smooth projective) surface, and let $E = \pi^{-1}(P)$ be the exceptional curve. Then $E \cong \mathbb{P}^1$ and $E^2 = -1$. We show that if a curve E on a (smooth projective) surface X has the properties that $E \cong \mathbb{P}^1$ and $E^2 = -1$, then E is the exceptional curve of the blow up of a point on a (smooth projective) surface.

Theorem 6.9. *(Castelnuovo's Contraction Theorem) If E is a curve on a smooth projective surface X such that $E \cong \mathbb{P}^1$ and $E^2 = -1$, then there exists a morphism $f : X \to X_0$ to a smooth projective surface X_0 such that $f(E) = P$ is a point and $f : X \to X_0$ is the blow up of P on X_0.*

Proof. (Sketch) We first want to find a line bundle $\mathcal{L}$ on X which is generated by global sections (hence induces a morphism $f : X \to \mathbb{P}^N_k$), is very ample when restricted to $X \setminus E$ (thus the restriction of f to $X \setminus \mathbb{P}^1$ is an immersion) and has the property that $f(E)$ is a point.

Let H be a very ample divisor on X such that $\mathrm{H}^1(X, \mathcal{L}(H)) = 0$; for example, a sufficiently high multiple of a very ample divisor [2, III, Theorem 5.3]. Then $H.E = m > 0$ and for $M = H + mE$ we have $M.E = 0$. Set $\mathcal{M} = \mathcal{L}(H + mE)$; we will show that the line bundle $\mathcal{M}$ defines a morphism $X \to \mathbb{P}^N_k$ with the desired properties.

For all $0 \leq i \leq m$, tensoring the exact sequence of line bundles $0 \to \mathcal{L}(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$ with $\mathcal{L}(H + iE)$ yields the exact sequence

$$0 \to \mathcal{L}(H + (i-1)E) \to \mathcal{L}(H + iE) \to \mathcal{O}_E \otimes \mathcal{L}(H + iE) \to 0.$$

Here $\deg(\mathcal{O}_E \otimes \mathcal{L}(H + iE)) = \#(E \cap (H + iE)) = m - i$ by Lemma 4.5 and the given isomorphism $E \cong \mathbb{P}^1_k$ maps $\mathcal{O}_E \otimes \mathcal{L}(H + iE)$ to $\mathcal{O}_{\mathbb{P}^1_k}(m - i)$. Consider the long exact sequence in cohomology

$$\begin{aligned} 0 \to \mathrm{H}^0(X, \mathcal{L}(H + (i-1)E) \to \mathrm{H}^0(X, \mathcal{L}(H + iE)) \to \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m - i)) \\ \to \mathrm{H}^1(X, \mathcal{L}(H + (i-1)E)) \to \cdots \end{aligned}$$

We claim that $\mathrm{H}^1(X, \mathcal{L}(H + iE)) = 0$ for all $1 \leq i \leq m$. By Serre Duality (or more elementary methods)

$$\dim_k \mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m - i)) = \dim_k \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2 + i - m))$$

which is zero for $i \leq m$. Hence, if $\mathrm{H}^1(X, \mathcal{L}(H + (i-1)E) = 0$, then $\mathrm{H}^1(X, \mathcal{L}(H + iE)) = 0$ for $1 \leq i \leq m$; by our choice of H this is true for $i = 1$, hence for $i \leq m$. Thus for $1 \leq i \leq m$ there is the exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{L}(H+(i-1)E)) \xrightarrow{\cdot t} \mathrm{H}^0(X, \mathcal{L}(H+iE)) \to \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m-i)) \to 0,$$

where t is an element of $\mathcal{L}(H)$ which vanishes on E. If $\{s_0, \ldots, s_n\}$ is a basis of $\mathrm{H}^0(X, \mathcal{L}(H))$, then $H + E$ has sections $ts_0, \ldots, ts_n$ plus the pullbacks of the m sections of $\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m - 1))$. Thus $\mathrm{H}^0(X, \mathcal{L}(H + mE))$ has a basis of global sections $\{t^m s_0, \ldots, t^m s_n, t^{k-1} v_0^{(k-1)}, \ldots, t^{k-1} v_{k-1}^{(k-1)}, \ldots t v_0^{(1)}, t v_1^{(1)}, v\}$, where the $v_0^{(r)}, \ldots, v_r^{(r)}$ are the pullbacks of $r + 1$ sections of $\mathcal{O}_{\mathbb{P}^1_k}(r)$ and v is the pullback of a section of $\mathcal{O}_{\mathbb{P}^1_k}$; in particular, all of these sections, except for the last one v, vanish on E. It follows that $\mathcal{M}$ is generated by global sections and is very ample outside of E, i.e. $\mathcal{M}$ induces a morphism $f_1 : X \to \mathbb{P}^N_k$ whose restriction to $X \setminus E$ is an immersion $X \setminus E \to \mathbb{P}^N_k$. Furthermore, since v is the only section in the basis of $\mathrm{H}^0(X, \mathcal{L}(H + kE))$, which does not vanish on E, $f_1(E) = P_1$ is a point. Set $X_1 = f_1(X)$.

Let $\pi : X_0 \to X$ be the normalization of X_1 [2, II, Exercise 3.8]. Since X is smooth, it is normal and the universal property of the normalization gives a morphism $f : X \to X_0$ such that $f_1 = \pi \circ f_0$. Because E is irreducibel, $f(E) = P$ is still a point and the restriction of f to $X \setminus E$ induces an isomorphism $X \setminus E \to X_0 \setminus P$, where X_0 is a normal variety and smooth outside of P.

It remains to show that X_0 is smooth at P. Since X_0 is normal and f is birational, we have $f_*\mathcal{O}_X \cong \mathcal{O}_{X_0}$ and one can use the Theorem on Formal Functions [2, III, Theorem 11.1] to show that $\widehat{\mathcal{O}}_{X_0,P} \cong k[[x,y]]$, which is a regular local ring. It follows that $\mathcal{O}_{X_0,P}$ is a regular local ring [2, I, Theorem 5.4A], i.e. P is a smooth point on X_0.

To conclude we use Lemma 6.8: By construction the morphism $f : X \to X_0$ is a birational morphism with $f(E) = P$ and $n(f) = 1$. Hence $f : X \to X_0$ must be the blow up of the point P on X_0. □

In the classification of smooth projective surfaces we want to specify within each birational equivalence class of surfaces, one which is as canonical as possible. Since one can always blow up a point without changing the birational equivalence class, there is never a unique smooth projective model (as in the case of curves). However, there are always relatively minimal models as follows:

Definition 6.10. A smooth projective surface X is a relatively minimal model for its function field, if every birational morphism $X \to X'$ to another smooth projective surfaces is an isomorphism. If X is the unique relatively minimal models in its birational equivalence class, then X is a minimal model.

Theorem 6.11. *Every smooth projective surface admits a birational morphism to a relatively minimal model.*

Proof. It follows from Lemma 6.8 and Theorem 6.9 that X is a relatively minimal model if and only if it contains no exceptional curves of the first kind. Given X, if X is already a relatively minimal model, set $f = \mathrm{id} : X \to X$. If not, X has an exceptional curve E of the first kind and blowing down E yields a birational morphism $X = X_0 \to X_1$. Iteration yields a sequence of blow downs $X = X_0 \to X_1 \to \cdots \to X_n$, and we need to show this process terminates. In each step $X_i \to X_{i+1}$ the contraction of the exceptional curve decreases the integral rank of the Neron-Severi group by 1, hence $n \leq \mathrm{Rang}_{\mathbb{Z}} \mathrm{NS}(X)$, which is finite, since $\mathrm{NS}(X)$ is a finitely generated abelian group (cp. [2, V, Exercise 1.7]). □

Remarks 6.12. (a) It is not true that a smooth projective surface contains only finitely many exceptional curves of the first kind. Generally sucessive blow ups of points can generate more than the expected exceptional curves of the first kind. For example, if X is the sucessive blow up of two distinct points $P \neq Q$ on $\mathbb{P}^2_k$, X contains the two exceptional curves of the first kind which are the preimages of P and Q, but it also contains the strict transform $\widetilde{L}$ of the line L through P and Q in $\mathbb{P}^2_k$. By Corollary 5.5 $g(\widetilde{L}) = 0$, i.e. $\widetilde{L} \cong \mathbb{P}^1_k$, and as in Example 4.7(e) we have $\widetilde{L}^2 = -1$. It can be shown

[2, V, Exercise 4.15] that if X is the blow up of 9 general points on $\mathbb{P}^2_k$, then X contains infinitely many exceptional curves of the first kind.

(b) A classical result of Zariski says that except for rational ($X \sim \mathbb{P}^2_k$) and ruled surfaces ($X \sim \mathbb{P}^1_k \times C$, where C is a smooth curve of positive genus) every smooth projective surface is birational to a unique minimal models. For rational and ruled surfaces one can describe the relatively minimal models explicitely.

7. Classification of Surfaces

In the case of curves every birational equivalence class contains a unique smooth projective model. The genus of a smooth projective curve C is a numerical invariant which can assume all non-negative integers $g(C) \geq 0$. We have seen that $g(C) = 0 \Leftrightarrow C \cong \mathbb{P}^1_k$, $g(C) = 1 \Leftrightarrow C$ is an elliptic curve and $g(C) \geq 2 \Leftrightarrow C$ is a curve of general type. We reformulate this classification:

Definition 7.1. Let X be a smooth projective variety, and let K_X be a canonical divisor. The Kodaira dimension of X is defined as the integer

$$\kappa(X) = \operatorname{tr.deg}_k(\oplus_{n\geq 0} \mathrm{H}^0(X, \mathcal{L}(nK_X))) - 1.$$

NB. Let $n \geq 0$ and assume $l(nK_X) = d_n \neq 0$. Then $|nK_X|$ defines a canonical rational map $X \dashrightarrow \mathbb{P}^{d_n-1}_k$ and the Kodaira dimension $\kappa(X)$ is the largest dimension of the image of these maps for $n \geq 1$ and $\kappa(X) = -1$ (or $\kappa(X) = -\infty$), if $|nK_X| = \emptyset$ for all $n \geq 1$. In particular, if $\dim X = d$, we have for the Kodaira dimension $\kappa(X) \in \{-1, 0, 1, \ldots, d\}$. Equivalenty, the plurigenera of X are the non-negative integers $P_n = P_n(X) = \dim_k \mathrm{H}^0(X, \mathcal{L}(nK_X))$, where $n \geq 0$ and $\kappa = \kappa(X)$ is the smallest number such that P_n/n^κ is bounded, and $\kappa(X) = -1$ (or $-\infty$), if $P_n = 0$ for all $n \geq 1$. The plurigenera P_n for $n \geq 0$ and the Kodaira dimension $\kappa(X)$ are birational invariants.

Examples 7.2. (a) Let C be a smooth projective curve. By definition $P_1(C) = g(C)$. If $g(C) = 0$, then $C \cong \mathbb{P}^1_k$ and $P_n(C) = P_n(\mathbb{P}^1_k)$ for all $n \geq 0 \Leftrightarrow \kappa(C) = \kappa(\mathbb{P}^1_k) = -1$, because $\mathcal{L}(nK_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1_k}(-2n)$ and $\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2n)) = 0$ for $n \geq 1$; thus $g(C) = 0 \Leftrightarrow \kappa(C) = -1$. If $g(C) = 1$, then $C = E$ is an elliptic curve, $\mathcal{L}(nK_E) = \mathcal{O}_E$ and $P_n(E) = 1$ for all $n \geq 1$, i.e. $\kappa(E) = 0$. In case $g(C) \geq 2$, i.e. for a curve of general type, $\kappa(C) = 1$.

(b) If $X = C \times D$ is a product of smooth projective curves and one writes $\kappa(X) = -\infty$, if $P_n(X) = 0$ for all $n \geq 1$, we have the formula $\kappa(X) = \kappa(C) + \kappa(D)$ and with suitable choices of curves C and D we can produce surfaces $X = C \times D$ of Kodaira dimension $\kappa(X) \in \{-\infty, 0, 1, 2\}$. Note that for a product $\mathbb{P}^1_k \times E$ with E an elliptic curve $\kappa(\mathbb{P}^1_k \times E) = -\infty$, but $\mathbb{P}^1_k \times E$ is not rational.

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The classification of surfaces uses the Kodaira dimension, as well as the geometric genus $p_g = \dim_k \mathrm{H}^2(X, \mathcal{O}_X)$, the irregularity $q = \dim_k \mathrm{H}^1(X, \mathcal{O}_X)$, and the arithmetic genus $p_a = p_g - q$. The following Theorem summarizes this classification:

Theorem 7.3. *(Classification by Kodaira dimension) Let X be a smooth projective surfaces over an algebraically closed field of characteristic 0.*

(a) $\kappa(X) = -\infty \Leftrightarrow X$ *is either rational or ruled.*

(b) $\kappa(X) = 0$. *Then a surface in this class must be one of the following:*
   (i) *a $K3$-surface, which is defined as a surface with $K_X = 0$ and $q = 0$. These have $p_a = p_g = 1$,*
   (ii) *an Enriques surface, which has $p_a = p_g = 0$ and $2K_X = 0$,*
   (iii) *a 2-dimensional abelian variety; then $p_a = -1$ and $p_g = 1$,*
   (iv) *a hyperelliptic surface X, which is a surfaced fibered over $\mathbb{P}^1_k$ by a pencil of elliptic curves.*

(c) $\kappa(X) = 1$. *Then X is an elliptic surface, i.e. a surface X with a morphism $\pi : X \to C$ to a curve C such that almost all fibers of π are smooth elliptic curves.*

(d) $\kappa(X) = 2 \Leftrightarrow$*for some $n > 0$ the complete linear system $|nK_X|$ determines a birational morphism from X onto its image in $\mathbb{P}^N_k$. These surfaces are surfaces of general type.*

8. Rational Surfaces

We consider rational surfaces, i.e. smooth projective surfaces X which are birational to $\mathbb{P}^2_k$, $X \sim \mathbb{P}^2_k$. First examples of rational surfaces are $\mathbb{P}^2_k$, linear hypersurfaces in $\mathbb{P}^3_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$, or equivalently quadric hypersurfaces in $\mathbb{P}^3_k$. If $X \sim \mathbb{P}^2_k$, we have for the birational invariants q, p_g and P_2:

$$\begin{array}{rcccl} q(X) = q(\mathbb{P}^2_k) & = & \dim_k \mathrm{H}^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) & = & 0, \\ p_g(X) = p_g(\mathbb{P}^2_k) & = & \dim_k \mathrm{H}^2(X, \mathcal{O}_{\mathbb{P}^2_k}) & = & \dim_k \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-3)) = 0, \\ P_1(X) = P_1(\mathbb{P}^2_k) & = & \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \omega_{\mathbb{P}^2_k}) & = & \dim_k \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-3)) = 0. \\ P_2(X) = P_2(\mathbb{P}^2_k) & = & \dim_k \mathrm{H}^2(\mathbb{P}^2_k, \omega^{\otimes 2}_{\mathbb{P}^2_k}) & = & \dim_k \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-6)) = 0. \end{array}$$

Since for a smooth projective $C \sim \mathbb{P}^1_k \Leftrightarrow g(C) = \dim_k \mathrm{H}^1(C, \mathcal{O}_c) = 0$, one might expect that for a smooth projective surface we have the equivalence

$$X \sim \mathbb{P}^2_k \Leftrightarrow q = \dim_k \mathrm{H}^1(X, \mathcal{O}_X) = 0 \text{ and } p_g = \dim_k \mathrm{H}^2(X, \mathcal{O}_X) = 0.$$

This is false.

Example 8.1. Let $X \subseteq \mathbb{P}^3_k$ be the quintic Fermat hypersurface, i.e. the (smooth) surface defined by the equation $T_0^5 + T_1^5 + T_2^5 + T_3^5 = 0$. The cyclic group $G = \langle \rho \rangle$ of order 5 acts on the points of X by $(X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : \rho X_2 : \rho^2 X_2 : \rho^3 X_3)$ and the (classical) Godeaux surface is the (smooth projective) surface $Y = X/G$. One can show that $q(Y) = 0 = p_g(Y)$ but Y is not rational: For example, any smooth projective rational variety must be simply connected; however $\pi(Y) \cong G$.

For any rational surface X we have $P_1(X) = P_2(X) = 0$ and therefore $\kappa(X) = -\infty$. The classification Theorem 7.3(a) states that such a surface is either rational or ruled. We have the following more precise result:

Theorem 8.2. *(Castelnuovo) Let X be a smooth projective surface with $q(X) = 0 = P_2(X)$. Then X is rational.*

Example 8.3. Let $i : X \to \mathbb{P}^3_k$ be a smooth hypersurface of degree d. The short exact sequence of sheaves $0 \to \mathcal{L}(-d) \to \mathcal{O}_{\mathbb{P}^3_k} \to \mathcal{O}_X \to 0$ on $\mathbb{P}^3_k$ induces a long exact sequence in cohomology, where by the standard cohomology vanishing on $\mathbb{P}^3_k$, $\mathrm{H}^1(\mathbb{P}^3_k, i_*\mathcal{O}_X) = 0$. Since by [2, III, Lemma 2.10] $\mathrm{H}^1(X, \mathcal{O}_X) \cong \mathrm{H}^1(\mathbb{P}^3_k, i_*\mathcal{O}_X) = 0$ every such hypersurface X has $q(X) = 0$. Furthermore, $\omega_X \cong \mathcal{O}_X(d-n-1)$, where $\mathcal{O}_X(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^3_k}(1)$ to X. It follows that if $d \leq n$, then $P_m = 0$ for all $m \geq 1$ and X is rational by Theorem 8.2. If $d = n+1$, then $\omega_X \cong \mathcal{O}_X$ and $P_m = 1$ for all $m \geq 1$, and if $d \geq n+2$, then $P_1(X) > 0$; i.e. these hypersurfaces cannot be rational.

Recall that a smooth projective variety of dimension n is unirational, if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$. Hence a variety is rational (resp. unirational) if its function field is a purely transcendental field extension (resp. is contained in a purely transcendental field extension) of k.

If C is a smooth projective unirational curve, the corresponding dominant rational map $f : \mathbb{P}^1_k \dashrightarrow C$ is a surjective morphism, which implies that $g(C) = 0$, thus C is rational. The question wether or nor every unirational variety is rational is Lüroth's problem. For surfaces Theorem 8.2 implies:

Corollary 8.4. *Every smooth projective unirational surface is rational.*

Proof. (of Corollary of 8.4) If X is smooth projective unirational surface, there is a dominant rational map $\mathbb{P}^2_k \dashrightarrow X$. By the Factorization Theorem 6.4 this map can be factored as a finite sequence of monoidal transformations and their inverses. Since q and P_2 are invariant under monoidal transformation we have $q(X) = 0 = P_2(X)$ and it follows from Castelnuovo's Theorem 8.2 that X is rational. □

To sketch the proof of Theorem 8.2 we start with the following Theorem proved by Noether-Enriques:

Theorem 8.5. *(Noether-Enriques) Let X be a smooth projective surface and $\pi : X \to C$ a morphism to a smooth projective curve C. If $P \in C$ is a point such that $\pi^{-1}(P) \cong \mathbb{P}^1_k$, then there exists an open subset $U \subseteq C$ containing P such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{P}^1_k$ over U.*

NB. If $C \cong \mathbb{P}^1_k$, this Theorem implies that $X \sim \mathbb{P}^1_k \times \mathbb{P}^1_k$, hence $X \sim \mathbb{P}^2_k$.

Proof. See [1, Theorem III.4]. □

The above Theorem gives a criterion for when a smooth projective surface X is rational. To construct a morphism $\pi : X \to \mathbb{P}^1_k$ as in Theorem 8.5 one considers morphisms coming from suitable divisors. Crucial and not obvious is the following Lemma:

Lemma 8.6. *Let X be a minimal smooth projective surface (that is, X is a smooth projective surface which does not contain curves $C \cong \mathbb{P}^1_k$ with $C^2 = -1$) with $q(X) = 0 = P_2(X)$. Then there is a curve C on X with $C \cong \mathbb{P}^1_k$ and $C^2 \geq 0$.*

Proof. (of Theorem 8.2, assuming Lemma 8.6) If X is a smooth projective surface, Theorem 6.11 shows that X admits a birational morphism $X \to X_n$ to a minimal smooth projective surface X_n. Since $q(X_n) = 0$ and $P_2(X_n) = 0$ it suffices to prove the Theorem for a minimal smooth projective surface X_n (then $X \sim X_n \sim \mathbb{P}^2_k$) and we may assume $X = X_n$ is minimal. By Lemma 8.6 there is a curve C on X with $C \cong \mathbb{P}^1_k$ and $C^2 \geq 0$. Tensoring the exact sequence $0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ with $\mathcal{L}(C)$ yields the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{O}_C \otimes \mathcal{L}(C) \to 0.$$

Since $q(X) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X) = 0$ we see that

$$\begin{aligned} \dim_k \mathrm{H}^0(X, \mathcal{L}(C)) &= \dim_k \mathrm{H}^0(X, \mathcal{O}_X) + \dim_k \mathrm{H}^0(X, \mathcal{O}_C \otimes \mathcal{L}(C)) \\ &= 1 + \dim_k \mathrm{H}^0(X, \mathcal{O}_C \otimes \mathcal{L}(C)). \end{aligned}$$

The Riemann-Roch Theorem for curves 3.3 implies that

$$\dim_k \mathrm{H}^0(C, \mathcal{O}_C \otimes \mathcal{L}(C)) = \deg(\mathcal{O}_C \otimes \mathcal{L}(C)) + l(\omega_c - \mathcal{O}_C \otimes \mathcal{L}(C)) + 1 - g(C),$$

where $\deg(\mathcal{O}_C \otimes \mathcal{L}(C)) = C^2 \geq 0$ by Example 4.7(a) and $g(C) = 0$ since $C \cong \mathbb{P}^1_k$. Furthermore $\omega_C - \mathcal{O}_C \otimes \mathcal{L}(C) = \omega_C \otimes \mathcal{O}_C \otimes \mathcal{L}(-C) = \mathcal{O}_{\mathbb{P}^1_k}(-2) \otimes \mathcal{O}_C \otimes \mathcal{L}(-C)$ and $\deg(\mathcal{O}_{\mathbb{P}^1_k}(-2) \otimes \mathcal{O}_C \otimes \mathcal{L}(-C)) = -2 - \deg(\mathcal{O}_C \otimes \mathcal{L}(C) = -2 - C^2 < 0$, hence $l(\omega_C - \mathcal{O}_C \otimes \mathcal{L}(C)) = 0$, and therefore

$$l(\mathcal{L}(C)) = \dim_k \mathrm{H}^0(X, \mathcal{L}(C)) = 2 + C^2 \geq 2.$$

This means the projective space $|C| \cong \mathbb{P}_k^{l(\mathcal{L}(C))-1}$ has positive dimension and there is a $D \in |C|$ such that $C \neq mD$ for all $m \geq 0$. Let $|V|$ be the linear system generated by C and D an let $\pi : X \dashrightarrow \mathbb{P}^1_k$ be the induced rational map. If $|V|$ is base-point free, π is a morphism and has a fiber $\pi^{-1}(P) \cong \mathbb{P}^1_k$, so that by Theorem 8.5 X is rational. If $|V|$ has a base point P, let $\widetilde{\pi} : \widetilde{X} \to X$ be the blow up of P on X with exceptional curve E. Since P is a base point, it lies on every divisor in $|V|$. Since C is a smooth curve $\widetilde{\pi}^*(C) = \widetilde{C} + E$ and $\widetilde{C} \cong C \cong \mathbb{P}^1_k$, i.e. $\widetilde{X}$ is rational by Theorem 8.5. Since the set where π is not defined is a finite set of closed points one obtains after a finite number of blow ups a birational morphism from a rational surface to X. □

9. Ruled Surfaces

In this section we mainly follow Beauville, see [1, Chapter III].

Definition 9.1. A smooth projective surface X is ruled (over C), if there is a smooth projective curve C such that $X \sim C \times \mathbb{P}^1_k$. If $C = \mathbb{P}^1_k$, then X is rational.

Examples 9.2. (a) If C is a smooth projective curve, then $C \times \mathbb{P}^1_k$ is ruled.
(b) Let C be a smooth projective curve and let $\mathcal{E}$ be a locally free sheave of rank 2 on C. Then the associated projective bundle $\mathbb{P}_C(\mathcal{E})$ is a ruled surface (more precisely, let $\mathcal{S} = \oplus_{d>0} S^d(\mathcal{E})$ be the symmetric algebra associated with $\mathcal{E}$ [2, II, Exercise 5.16]. Then $\mathbb{P}_C(\mathcal{E}) = \mathrm{Proj}(\mathcal{S})$ [2, pg. 162]. There is a projection morphism $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ such that if $\mathcal{E}$ is free on an open set $U \subseteq C$, then $\pi^{-1}(U) \cong \mathbb{P}^1_U = \mathbb{P}^1_k \times U$; hence $\mathbb{P}_C(\mathcal{E})$ is a ruled surface).

Definition 9.3. A smooth projective surface X is geometrically ruled (over C), if there is a surjective morphism $\pi : X \to C$ to a smooth projective curve C such that each fiber is isomorphic to $\mathbb{P}^1_k$.

Proposition 9.4. *Every geometrically ruled surface is ruled.*

Proof. This follows directly from the Noether-Enriques Theorem 8.5: For a geometrically ruled surface X with morphism $\pi : X \to C$ and $P \in C$, by definition $\pi^{-1}(P) \cong \mathbb{P}^1_k$. By Theorem 8.5 there is an open set $U \subset C$ containing P such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1_k$. In particular, the fibration $\pi : X \to C$ is locally trivial and X is ruled. $\square$

Lemma 9.5. *Let X be a ruled surface over C. Then $q(X) = g(C)$ and $P_m(X) = 0$ for all $m \geq 1$; in particular $\kappa(X) = -\infty$.*

Proof. We may assume $X = C \times \mathbb{P}^1_k$ for a smooth projective curve C. Let $p_1 : X \to C$ and $p_2 : X \to \mathbb{P}^1_k$ be the projections. There is an isomorphism $\Omega^1_X \cong p_1^*\Omega^1_C \oplus p_2^*\Omega^1_{\mathbb{P}^1_k}$ which implies $\mathrm{H}^1(X, \Omega^1_X) \cong \mathrm{H}^0(C, \omega_C) \oplus \mathrm{H}^0(\mathbb{P}^1_k, \omega_{\mathbb{P}^1_k})$. Hence $q(X) = g(C) + g(\mathbb{P}^1_k) = g(C)$.

For any two locally free sheaves $\mathcal{F}$ on C and $\mathcal{G}$ on $\mathbb{P}^1_k$ there is an isomorphism $\mathrm{H}^0(X, \mathcal{F}) \otimes \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{G}) \to \mathrm{H}^0(X, p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$. In particular, this implies that $\mathrm{H}^0(C, \omega_C^{\otimes m}) \otimes \mathrm{H}^0(\mathbb{P}^1_k, \omega_{\mathbb{P}^1_k}^{\otimes m}) \cong \mathrm{H}^0(X, \omega_X^{\otimes m}) = 0$ for $m \geq 1$. $\square$

Proposition 9.6. *(a) Let X be a geometrically ruled surface over C. Then there is a locally free sheaf $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}_C(\mathcal{E})$ over C.*
(b) If $\mathcal{E}$ and $\mathcal{E}'$ are two locally free sheaves of rank 2 on C, then $\mathbb{P}_C(\mathcal{E}) \cong \mathbb{P}_C(\mathcal{E}')$ if and only if $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on C.

Proof. If X is geometrically ruled over C, Proposition 9.4 implies that the fibration $\pi : X \to C$ is locally trivial. The isomorphism classes of such bundles are bijection with the set $\mathrm{H}^1(C, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of (non-abelian) groups $\mathcal{G} = \mathcal{GL}(2, \mathcal{O}_C)$ of invertible 2×2 matrices with entries in $\mathcal{O}_C$. The Proposition follows from this, for details see [1, Proposition III.7]. $\square$

Theorem 9.7. *Let C be a smooth projective curve with $g(C) > 0$. Then the relatively minimal models of $C \times \mathbb{P}^1_k$ are the geometrically ruled surfaces, i.e. the projective bundles $\mathbb{P}_C(\mathcal{E})$.*

Lemma 9.8. *Let X be a minimal smooth projective surface and $\pi : X \to C$ a morphism with generic fiber isomorph to $\mathbb{P}^1_k$. Then X is a geometrically ruled surface over C (with respect to π).*

Proof. □

For the proof of Theorem 9.7 we need the following weaker version of the Factorization Theorem 6.4: Let $\phi : S \dashrightarrow X$ be a rational map from a smooth projective surface to a smooth projective variety. Then there exists a smooth projective surface S', a morphism $f_n : S' \to S$ which is the composite of n monoidal transformations and a morphism $f : S' \to X$ such that $\phi \circ f_n = f$; see [1, Theorem II.7].

Proof. (of Theorem 9.7). Let $\pi : X \to C$ be a geometrically ruled surface. To show X is minimal it suffices to show that X contains no exceptional curves E of the first kind. Since any fiber F of π has $F^2 = 0$ no fiber can be an exceptional curve. If E is an exceptional curve of the first kind, then E is not a fiber and therefore $f(E) = C$. Since then $g(E) \geq g(C)$ and $g(E) = 0$, it follows that $g(C) = 0$, hence $C \cong \mathbb{P}^1_k$, which is a contradiction. Hence any such geometrically ruled surface is minimal.

Assume X is a relatively minimal model of $C \times \mathbb{P}^1_k$. We show X is geometrically ruled over C. Let $\phi : X \dashrightarrow C \times Y$ be the rational map, $q : C \times \mathbb{P}^1 \to \mathbb{P}^1_k$ the projection, and consider $q \circ \phi : X \dashrightarrow C \times \mathbb{P}^1_k$. By the above there is a smooth projective surface X', together with a finite number n of monoidal transformations $f_n : X' \to X$ and a morphism $f : X' \to C \times \mathbb{P}^1_k$ such that $f_n \circ (q \circ \phi) = f$. Let n be the minimal number of monoidal transformations such that there is a such a diagram and assume $n > 0$. Let E be the exceptional curve on X' which is contracted to a point under the first monoidal transformation. Then $f(E)$ is a point (else $f(E) = C$ which implies C is rational). By a universal property of a monoidal transformation (if $\widetilde{X} \to X$ is a moinoidal transformation with exceptional curve $E$and $f : \widetilde{X} \to Y$ is any morphism to a variety that contracts E to a point, then f factors through X, see [1, Remarks II.13]) f factors through X which contradicts the minimality of n. Hence $n = 0$ and $f = q \circ \phi : X \to C$ is a morphism with generic fiber isomorphic to $\mathbb{P}^1_k$. By Lemma 9.8 X is geometrically ruled (with respect to $f : X \to C$). □

The above proof uses repeatedly the fact that C is not rational and does not apply to rational surfaces. The study of geometrically ruled surface $\pi : X \to C$ over a fixed arbitrary smooth projective curve C is by Proposition 9.6 equivalent to the study of the projective bundles associated to locally free sheaves of rank 2 on C.

If $\mathcal{E}$ is such a locally free sheaf on a smooth projective curve C, then $\mathcal{E}$ always fits in a short exact sequence of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0,$$

where $\mathcal{L}$ and $\mathcal{M}$ are line bundles on C; however, this short exact sequence is not split in general. If $C = \mathbb{P}^1_k$, we have the following:

Proposition 9.9. *Let $\mathcal{E}$ be a locally free sheaf of rank 2 on $\mathbb{P}^1_k$. Then*

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(n) \text{ for some } n \geq 0.$$

In particular, if X is a geometrically ruled surface over $\mathbb{P}^1_k$, then

$$X \cong \mathbb{P}_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(n)) \text{ for some } n \geq 0.$$

Proof. [1, Proposition III.15]. □

Definition 9.10. The Hirzebruch surface (or the rational scroll) $\mathbb{F}_n$ $(n \geq 0)$ is the geometrically ruled surface over $\mathbb{P}^1_k$ associated to locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{P^1_k}(n))$ on $\mathbb{P}^1_k$.

The surface $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$, hence rational. On the other hand $\mathbb{F}_1$ is isomorphic to the blow up of a point on $\mathbb{P}^1_k$, hence rational but not minimal. More generally, one can show (using the intersection pairing on a geometrically ruled surface):

Proposition 9.11. *The surfaces $\mathcal{F}_n$ are minimal, unless $n = 1$. Furthermore, $\mathcal{F}_n \cong \mathcal{F}_m$ if and only if $n = m$.*

Proof. [1, Proposition IV.1(iii)].

□

Finally, we have

Theorem 9.12. *Let X be a minimal rational surface. Then $X \cong \mathbb{P}^2_k$ or $X \cong \mathbb{F}_n$ for some $n \neq 1$.*

Proof. [1, Theorem V.10]. □