Note on the Quantization of Massless and Massive Vector Fields

徐鹤峰 PB19020673

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1 Introduction

In the class of Quantum Field Theory I, we've learned about the canonical quantization of real scalar field, complex scalar field and Dirac fields.

However we encountered some problems when we try to apply this technique to spin one particles, say, vector fields.

In this note, I try to solve some of them with several methods to fulfill the structure of the quantization.

As we know, particles can be considered as an irreducible representation of the Poincaré group. Due to the requirement of invariants, we expect the transformation of particles to be unitary. Wigner proved that any irreducible positive representation can be massed by two quantum numbers m and spin J.

To consider spin for $J = 1$ of particles, we embed this irreducible representation into vector fields with space-time as an indicator A^μ medium, which is an uncompactive representation of the poincaré group. we will find the difference between the two representations in degrees of freedom, A^μ there are 4 degrees of freedom, while a mass spin one particle has only 3 degrees of freedom and a massless spin 1 particle has only 2 degrees of freedom, and this redundant degree of freedom will bring canonical symmetry and some problems.

In the following, we use vector field to refer to the spin one particle field. Let's first consider quantization of massless vector fields.

2 Quantization of massless vector fields

2.1 Some Presets to be Done

WLOG, we want to simplify the problem, so we only talk about the free electromagnetic field here as an example.

Here we have the Lagrangian as

$$\mathcal{L}_{Maxwell} = -\frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2.1}$$

Here $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Then we have the conjugate momentum of A^μ

$$\begin{aligned} \pi^0 &= \frac{\partial \mathcal{L}}{\dot{A}_0} \\ &= 0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} \pi^i &= \frac{\partial \mathcal{L}}{\dot{A}_0} \\ &= -F^{0i} \\ &= E^i \end{aligned} \tag{2.3}$$

Here for the electromagnetic field in vaccum space,

$$\nabla \cdot \mathbf{E} = 0 \tag{2.4}$$

2.1.1 Gauge?

Here, based on the equation above, we can have a classic result that

$$[A^0(x), \pi^0(y)]|_{x^0=y^0=0} = 0 \tag{2.5}$$

which is obviously incorrect. So we try to consider some quantum gauge. Easily, we pick up lorentz gauge as

$$\partial_\mu A^\mu = 0 \tag{2.6}$$

However, if we perform the canonical quantization with this gauge, we will meet 2 main problems as we did in class:

- Positive Mode:

  We have the canonical commutator of the annilators as

$$[a_{o\mathbf{P}}, a^{\dagger}_{o\mathbf{P}}] = -(2\pi)^3\delta^{(3)}(\mathbf{P} - \mathbf{P'}) \tag{2.7}$$

  Then we have

$$\begin{aligned} |a^{\dagger}_{o\mathbf{P}}|0\rangle|^2 &= \langle 0|[a_{o\mathbf{P}}, a^{\dagger}_{o\mathbf{P}}]|0\rangle \\ &= -(2\pi)^3\delta^{(3)}(0) < 0 \end{aligned} \tag{2.8}$$

  Which is obviously incorrect.

- Positive Definition of Energy

  Consider the Hamiltonian

$$\begin{aligned} H &= \int d^3x\mathcal{H} \\ &= -\frac{1}{2}d^3x(\dot{A}^{\mu}\dot{A}_{\mu} + \nabla A_{\nu}A^{\nu}) \\ &= \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{P}}(\sum_{i=1}^{3} a^{i\dagger}_{\mathbf{P}}a^{i}_{\mathbf{q}} - a^{0\dagger}_{\mathbf{P}}a^{0}_{\mathbf{q}}) \end{aligned} \tag{2.9}$$

  Here we used the commutator of annilators and ignored the ground state energy.

  Notice that the Hamiltonian is not always positive definite in this process, so we infer that this quantization is not correct.

So we pick another gauge, which is called the *weak Poincare Gauge* (or, say, *Gupta − Bleuler Gauge*)as

$$\langle\Psi|\partial_{\mu}A^{\mu}|\psi\rangle = 0 \tag{2.10}$$

Then we have

$$\begin{aligned} &\langle\Psi|\partial^{\mu}A_{\mu}|\psi\rangle = 0 \\ &\Rightarrow \langle\Psi|\partial^{\mu}A^{(+)}_{\mu}|\psi\rangle + \langle\Psi|\partial^{\mu}A^{(-)}_{\mu}|\psi\rangle = 0 \\ &\Rightarrow \langle\Psi|\partial^{\mu}A^{(+)}_{\mu}|\psi\rangle + \langle\Psi|\partial^{\mu}A^{(+)}_{\mu}|\psi\rangle^* = 0 \\ &\Rightarrow \langle\Psi|\partial^{\mu}A^{(+)}_{\mu}|\psi\rangle = 0 \end{aligned} \tag{2.11}$$

Then we have

$$
\begin{aligned}
&\partial^\mu A_\mu^{(+)}|\Psi\rangle = 0 \\
&\Rightarrow \int \frac{d^3\mathbf{P}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{P}}}\sum_{\lambda=0}^{3}\epsilon_{\mu\lambda}a_\mathbf{P}^\lambda\partial^\mu e^{-ipx}|\Psi\rangle = 0 \\
&\Rightarrow \int \frac{d^3\mathbf{P}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{P}}}\sum_{\lambda=0}^{3}(-ip^\mu)\epsilon_{\mu\lambda}a_\mathbf{P}^\lambda\partial^\mu e^{-ipx}|\Psi\rangle = 0
\end{aligned}
\tag{2.12}
$$

Then if we operate it on the momentum of photon

$$
p^\mu = (k, 0, 0, k) \tag{2.13}
$$

we have

$$
\begin{aligned}
&\int \frac{d^3\mathbf{P}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{P}}}k(a_\mathbf{P}^0 - a_\mathbf{P}^3)e^{-ipx}|\Psi\rangle = 0 \\
&\Rightarrow (a_\mathbf{P}^0 - a_\mathbf{P}^3)|\Psi\rangle = 0
\end{aligned}
\tag{2.14}
$$

Then we may conduct the canonical quantization with this gauge.

2.2 Canonical Quantization

The procedure is very similar to the canonical quantization of the Dirac fields, so please let me abbreviate most of the calculation and give the results directly:

The annilators can be expressed as :

$$
\begin{aligned}
a_\mathbf{P}^\lambda &= \delta_\mu^\nu g^{\sigma\lambda}(2\pi)^3\int d^3(x)\epsilon_{\sigma\nu}\Phi_{\mathbf{P}'}^*(s)(+i)\overleftrightarrow{\partial_0}A^\mu \\
a_\mathbf{P}^\lambda &= g^{\sigma\lambda}(2\pi)^3\int d^3(x)\epsilon_{\sigma\nu}\Phi_{\mathbf{P}'}^*(s)(+i)\overleftrightarrow{\partial_0}A^\mu
\end{aligned}
\tag{2.15}
$$

Similarly, we have

$$
a_\mathbf{P}^{\lambda\dagger} = \delta_\mu^\nu g^{\sigma\lambda}(2\pi)^3\int d^3(x)\epsilon_{\sigma\nu}\Phi_{\mathbf{P}'}^*(s)(-i)\overleftrightarrow{\partial_0}A^\mu \tag{2.16}
$$

Considering

$$
\Phi_\mathbf{P}(x) = \frac{1}{(2\pi)^3\sqrt{2E_\mathbf{P}}}e^{-ipx} \tag{2.17}
$$

Then we have:

$$
\begin{aligned}
a_{\sigma\mathbf{P}} &= g_{\sigma\sigma'}\int d^3\mathbf{x}\epsilon_\mu^{\sigma'}\frac{ie^{ipx}}{\sqrt{2E_\mathbf{P}}}(\dot{A}^\mu - iE_\mathbf{P}A^\mu) \\
a_{\sigma\mathbf{P}}^\dagger &= g_{\sigma\sigma'}\int d^3\mathbf{x}\epsilon_\mu^{\sigma'}\frac{ie^{ipx}}{\sqrt{2E_\mathbf{P}}}(\dot{A}^\mu + iE_\mathbf{P}A^\mu)
\end{aligned}
\tag{2.18}
$$

Then we have the commutator as

$$\begin{aligned} [a_{\sigma\mathbf{P}}, a^{\dagger}_{\lambda\mathbf{P}}] &= -g_{\sigma\lambda}(2\pi)^3\delta^{(3)}(\mathbf{P}-\mathbf{P'}) \\ [a_{\sigma\mathbf{P}}, a_{\lambda\mathbf{P'}}] &= [a^{\dagger}_{\sigma\mathbf{P}}, a^{\dagger}_{\lambda\mathbf{P'}}] = 0 \end{aligned} \tag{2.19}$$

2.3 Check Some Facts

With this new gauge and the result of the canonical quantization, we cna try to solve the problems we encountered in the previous section:

Firstly we have:

$$\begin{aligned} & a^0_{\mathbf{P}}|\Psi\rangle = a^3_{\mathbf{P}}|\Psi\rangle \\ & \Rightarrow \langle\Psi|a^{0\dagger}_{\mathbf{P}} = \langle\Psi|a^{3\dagger}_{\mathbf{P}} \\ & \Rightarrow \langle\Psi|a^{0\dagger}_{\mathbf{P}}a^0_{\mathbf{P}}|\Psi\rangle = \langle\Psi|a^{3\dagger}_{\mathbf{P}}a^3_{\mathbf{P}}|\Psi\rangle \end{aligned} \tag{2.20}$$

Then we can use it to act on the Hamiltonian:

$$\begin{aligned} \langle\Psi|H|\Psi\rangle &= \langle\Psi|\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{P}}(\sum_{i=1}^{3}a^{i\dagger}_{\mathbf{P}}a^i_{\mathbf{q}} - a^{0\dagger}_{\mathbf{P}}a^0_{\mathbf{q}})|\Psi\rangle \\ &= \langle\Psi|\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{P}}(\sum_{i=1}^{2}a^{i\dagger}_{\mathbf{P}}a^i_{\mathbf{q}})|\Psi\rangle \end{aligned} \tag{2.21}$$

Actually, we can have the Hamiltonian horizontally polarized as

$$H = \int\frac{d^3p}{(2\pi)^3}E_{\mathbf{P}}(\sum_{i=1}^{2}a^{i\dagger}_{\mathbf{P}}a^i_q) \tag{2.22}$$

Also we can do the same thing to the momentum, angular momentum and some other observariables. Here I will no longer talk about them.

3 Quantization of massive vector fields

Here we begin to talk about the canonical quantization of the massive vector fields.

3.1 Presets

For the modification of the Lagrangian, we insert a term for mass in the Lagrangian for the free fields, as:

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2AA^{\mu}A_{\mu} \tag{3.1}$$

Here the field is called *Proca* field. Then we have the equation of motion in this field as:

$$\begin{aligned}
&\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \\
&\Rightarrow -\partial_\mu F^{\mu\nu} - m_A^2 A^\nu = 0
\end{aligned} \tag{3.2}$$

Since the term of mass, the Lagrangian is no longer gauge invariance, so we have to:

$$\begin{aligned}
&\partial_\nu\partial_\mu F^{\mu\nu} + m_A^2\partial_\nu A^\nu = 0 \\
&\Rightarrow \partial_\mu\partial_\nu F^{\nu\mu} + m_A^2\partial_\nu A^\nu = 0 \\
&\Rightarrow \partial_\nu A^\nu = 0
\end{aligned} \tag{3.3}$$

By the transformation above, we notice that the massive vector fields are automatically satisfied with Lorentz gauge.

Use this to the motion equation, we have

$$\begin{aligned}
&\partial_\mu F^{\mu\nu} + m_A^2 A^\nu = 0 \\
&\Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m_A^2 A^\nu = 0 \\
&\Rightarrow \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) + m_A^2 A^\nu = 0 \\
&\Rightarrow \partial^2 A^\nu + m_A^2 A^\nu = 0 \\
&\Rightarrow (\partial^2 + m_A^2)A^\nu = 0
\end{aligned} \tag{3.4}$$

Notice that there are 3 independent degrees of freedom.

Then we have the canonical momentum:

$$\begin{aligned}
&\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = -F^{0i} \\
&\pi^0 = 0
\end{aligned} \tag{3.5}$$

3.2 Canonical Quantization

Here we have the conditions for canonical quantization for the 0, 1, 2, 3 distributions:

$$\begin{aligned}
&[A^i(x), \pi^j(y)]|_{x^0=y^0} = ig^{ij}\delta(3)((x) - (y)) \\
&[A^i(x), \pi^j(y)]|_{x^0=y^0} = 0 \\
&[A^i(x), \pi^j(y)]|_{x^0=y^0} = [\pi^\mu(x), \pi^\nu(y)]|_{x^0=y^0} = 0 \\
&[A^0(x), A^0(y)]|_{x^0=y^0} = 0 \\
&[A^0(x), A^i(y)]|_{x^0=y^0} = \frac{ig^{ij}}{m^2}\partial_{i,j}\delta^{(3)}(\mathbf{x} - textbfy) \\
&\pi^0 = 0
\end{aligned} \tag{3.6}$$

Then we expand the field operator as the planar wave:

$$A^\mu(x) = \int \frac{d^3\mathbf{P}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{P}}}\sum_{\lambda=0}^{3}\epsilon^\mu_\lambda(a^\lambda_\mathbf{P}e^{-ipx} + a^{\lambda\dagger}_\mathbf{P}e^{ipx}) \tag{3.7}$$

And with the Poincaré gauge, we have:

$$p_\mu\epsilon^\mu_\lambda = 0 \tag{3.8}$$

Then there are only 3 distributions of the polarized vector to be independent.

WLOG, let 1, 2, 3 distributions to be independent, then we have:

$$\begin{aligned}
\epsilon^\mu_1 &= (0,1,0,0)^T \\
\epsilon^\mu_2 &= (0,0,1,0)^T \\
\epsilon^\mu_3 &= (\frac{k}{m},0,0,\frac{\omega}{m})^T \\
\epsilon^\mu_0 &= (\frac{\omega}{m},0,0,\frac{k}{m})^T
\end{aligned} \tag{3.9}$$

Then we have

$$\begin{aligned}
&g^{rr}\epsilon^\mu_r\epsilon^\nu_r = g^{\mu\nu} \\
&\Rightarrow \sum_{r=1}^{3}\epsilon^\mu_r\epsilon^\nu_r = (-g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}) \\
&AND\ k_\mu\sum_{r=1}^{3}\epsilon^\mu_r\epsilon^\nu_r = (-k^\nu - \frac{k_\mu k^\mu k^\nu}{m^2}) = 0
\end{aligned} \tag{3.10}$$

Then we have the annilators as :

$$\begin{aligned}
a_{s\mathbf{P}} &= \int d^3\mathbf{x}\frac{-ie^{ipx}}{\sqrt{2E_\mathbf{P}}}(\epsilon_{s0}\frac{1}{m^2}\partial_j\pi^j - \frac{1}{m^2}\epsilon_{si}\partial^i\partial_j\pi^j - \epsilon_{si}\pi^i - iE_\mathbf{P}\epsilon_{s0}A^0 - i\epsilon_{si}E_\mathbf{P}A^i) \\
a^\dagger_{s\mathbf{P}} &= \int d^3\mathbf{x}\frac{ie^{ipx}}{\sqrt{2E_\mathbf{P}}}(\epsilon_{s0}\frac{1}{m^2}\partial_j\pi^j - \frac{1}{m^2}\epsilon_{si}\partial^i\partial_j\pi^j - \epsilon_{si}\pi^i + iE_\mathbf{P}\epsilon_{s0}A^0 + i\epsilon_{si}E_\mathbf{P}A^i)
\end{aligned} \tag{3.11}$$

Similarly, we have the commutators as

$$\begin{aligned}
&[a_{s\mathbf{P}}, a^\dagger_{s'\mathbf{P'}}] = -g_{ss'}(2\pi)^3\delta^{(3)}(\mathbf{P}-\mathbf{P'}) \\
&[a_{s\mathbf{P}}, a_{s'\mathbf{P'}}] = [a^\dagger_{s\mathbf{P}}, a^\dagger_{s'\mathbf{P'}}] = 0
\end{aligned} \tag{3.12}$$

Then we have finished the canonical quantization of the massive vector field.

3.3 Check Some Facts

Consider the energy:

We have the density of Hamiltonian as:

$$\begin{aligned}\mathcal{H} &= \pi^\mu \dot{A}_\mu - \mathcal{L} \\ &= -\frac{1}{2}(\dot{A}^\mu \dot{A}_\mu + \nabla A_\nu \cdot \nabla A^\nu) - \frac{1}{2} m_A^2 A_\mu A^\mu\end{aligned} \tag{3.13}$$

Then we have the hamiltonian as

$$\begin{aligned}H &= \int d^3x \mathcal{H} \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{P}}} \sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_r^\mu \epsilon_{s\mu}((E_{\mathbf{P}}^2 + \mathbf{P}^2)(a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^s + a_{\mathbf{P}}^r a_{\mathbf{P}}^{s\dagger}) + m_A^2(a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^s + a_{\mathbf{P}}^r a_{\mathbf{P}}^{s\dagger})) \\ &= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{P}}} \sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_r^\mu \epsilon_{s\mu}(E_{\mathbf{P}}^2)(a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^s + a_{\mathbf{P}}^r a_{\mathbf{P}}^{s\dagger}) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{3}\sum_{s=1}^{3} \epsilon_r^\mu \epsilon_{s\mu}(E_{\mathbf{P}}^2)(a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^s + a_{\mathbf{P}}^r a_{\mathbf{P}}^{s\dagger}) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{3}\sum_{s=1}^{3} \delta_{rs}(E_{\mathbf{P}}^2)(a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^s + a_{\mathbf{P}}^r a_{\mathbf{P}}^{s\dagger}) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{3} E_{\mathbf{P}} a_{\mathbf{P}}^{r\dagger} a_{\mathbf{P}}^r\end{aligned} \tag{3.14}$$

Here we ignore the ground energy of vaccum space and admitting the

$$E_{\mathbf{P}}^2 = m^2 + \mathbf{P}^2 \tag{3.15}$$

4 Conclusion

In this note, I tried to use canonical quantization to act on massive and massless vector fields, and to some degrees, we did solved the problems we had in the classic quantization procedure.

However this note is not so complete for the deadline is near. I may complete the details in the future.

5 References

- Peskin & Schroeder, An Introduction to Quantum Field Theory;
- Srednicki, Quantum Field Theory;
- Weinberg, Quantum Field Theory I.